# Approximation Algorithm

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**Lecture 52**

Lecture 52 : 3/2-Approximation Algorithm for Multiway Cut Contd.

Welcome. So, in the last class we have seen a randomized rounding based algorithm for multi way cut. The high level idea of the algorithm is we solve the LP and take a random radius r you picked uniformly at random between 0 ,1 and take a random permutation of 1 to k. And, iteratively from $\pi(1),\ldots,\pi(k-1)$, I picked all the points which are the which are within r ball of within r ball of $s_i$ and build those clusters and at the end whatever points are remaining I put them in see in the $\pi(k)$-th part and output the boundary edges. So, that is a very natural randomized rounding based algorithm.

So, today we will see the analysis of the algorithm. So, let us begin. This is multi way cut. So, we need a couple of lemmas let us begin the first lemma says that for any index $l\in[k]$ and any two vertices $u,v\in V$ the mod of $x_{u,l}-x_{v,l}$ this of course, is less than equal to $l_1$ norm, but actually it is even less than equal to half of $l_1$ norm less than equal to half of $l_1$ norm of $x_u-x_v$ proof.

So, what is $l_1$ norm of $x_u-x_v$, this is $\sum|x_{u,i}-x_{v,i}|$. Let us separate out the $l$ from here. this is $|x_{u,l}-x_{v,l}|+\sum|x_{u,i}-x_{v,i}|$ . Now, without loss of generality we can assume that $x_{u,l}\geq x_{v,l}$ if not we can rename them.

So, without loss of generality assume that $x_{u,l}\geq x_{v,l}$. Now, you see what is then $|x_{u,l}-x_{v,l}|$ this is just without mod because $x_{u,l}\geq x_{v,l}$ ok good. Now, $x_{u,l}$ you can write it as know we have to use the condition that $x_{u,l}$ belongs to the simplex that means, $\sum x_{u,i}=1$. So, from there you can write $x_{u,l}=1-\sum x_{u,i}$. Similarly, this is $1-\sum x_{v,l}$.

So, what we have $i\neq l$ $x_{v,l}-x_{u,l}$ ok. Now, this is less than equal to if I replace $x_e$ $x_{v,i}-x_{u,i}$ with a mod. So, this is less than equal to $\sum_{i\in[k],i\neq l}|x_{v,i}-x_{u,i}|$. Now, you see this is now come here. this is $l_1$ norm minus $|x_{v,l}-x_{u,l}|$ which is same as $|x_{v,l}-x_{u,l}|$.

So, let us write it there. So, this is $|x_u - x_v|_1 - |x_{u,l} - x_{v,l}|$. So, what we have bringing minus $x_{u,l} - x_{v,l}$ mod of it to the left hand side we get $|x_{u,l} - x_{v,l}|$ this is less than equal to $\frac{1}{2}|x_u - x_v|_1$ which is exactly what we need to prove ok. The second lemma gives an equivalent condition when some point belongs to the ball of radius r. So, you said that we say that for all vertex $u \in V$, u belongs to $B(s_i, r)$ which is set of all vertices $v \in V$ such that $l_1$ distance of $s_i$ minus V.

So, more precisely $l_1$ distance $s_i - x_v$ this is less than equal to $l_1$ distance less than equal to r. This is u belongs to ball of radius r centered at $s_i$ if and only if $1 - x_{u,i} \leq r$ proof. So, ok just to normalize we define the balls distance half times $l_1$ norm $l_1$ distance between $x_{u,i}$ and $x_{v,i}$. This is just to ensure that the ball of radius 1.

contains every point ball of radius 1 around say $s_i$ contains every point because the maximum distance $l_1$ distance between $s_i$ or between any two points is maximized when one point is $s_i$ and the other is $s_j$ and in that case the $l_1$ distance is 2. So, we multiply with half to make it 1 and in particular we define balls in this way. In without this typically when we define balls in analysis or in other context, then it means the all points of distance at most the radius from the centre, but here we have a extra factor of half. ok. So, u belongs to $B(s_i, r)$ ok if and only if from the definition if and only if half $l_1$ norm of $s_i - x_u$ this is less than equal to r ok.

Now, equivalently so, what is this from here what is $l_1$ norm this is $\frac{1}{2}\sum_{j=1}^{k}|x_{s_i j} - x_{u,j}|$ this is less than equal to r. Now, recall $s_i$ always belongs to the i-th component $C_i$ and hence $x_{s_i, j} = 0$ for all $j \neq i$ and $x_{s_i, i} = 1$. So, these are 0 and this is 1. So, putting these values we get this is if and only if.

$\frac{1}{2}\sum_{j \in [k], j \neq i}|-x_{u,j}|$ because $x_{u,j}$ takes only non-negative value this is $x_{u,j} + 1 - x_{u,i}$, this is less than equal to r. Now, this is if and only if $\sum x_{u,j} = 1$. So, this we can write it as $1 - x_{u,i}$ this is less than equal to r hence $1 - x_{u,i}$ is less than equal to r. So, this proves the lemma. Now, with this lemma lemmas in hand we now prove that the approximation guarantee of our algorithm is at most $\frac{3}{2}$ theorem.

Our algorithm has an approximation ratio of at most $\frac{3}{2}$ proof . So, let ALG be the random variable denoting the weight of the multi way cut output by the algorithm. So, what we will show is expectation of ALG is less than equal to $\frac{3}{4}\sum_{e=\{u,v\}\in E}|x_u^*-x_v^*|$, where$x^*$ is a normal solution let us write $x_u$ because there are no other x. So, these are optimal solutions x and the 1 of this is enough now because this is less than this is equal to $\frac{3}{2}$ times ok because the because x is an optimal solution and LP opt is less than equal to opt. So, this is less than equal to $\frac{3}{2}$ opt.

So, all we need to show is this this inequality. which we show now. So, we have seen many times that writing this ALG or random variable as sum of indicator random variables makes our life very easy when dealing with expectation of random variables. So, we follow that for every age $e=\{u,v\}$ we define a random variable $Z_u$ or an indicator random variable. for the event that the edge e belongs to the multi way cut output by the algorithm.

Then what we have is ALG. is nothing, but $\sum_{e\in E} w_e Z_e$. So, what is expectation of ELG expectation of ALG is expectation of $\sum_{e\in E} w_e Z_e$. Now, linear it using linearity of expectation which does not need this random variables to be independent or any relation because this random variables $Z_e$ need not be independent. So, this is $\sum_{e\in E} w_e E[Z_e]$.

Now, expectation of $Z_e$ is the probability that e belongs to the cut $\sum_{e\in E} w_e Prob[e\in the\,cut]$. Now, what we will show is that this probability that e belongs to the cut if edge $e=\{u,v\}$ then we will show that this probabilities at most $\frac{3}{4}|x_u-x_v|$. So, this is less than equal to $\frac{3}{4}\sum_{e\in E} w_e|x_u-x_v|$ and this is exactly what we need to show. So, all we need to show is this probability that e belongs to the cut.

So, here is the lemma which will finish the proof for all edge e probability that e belongs to the cut is less than equal to $\frac{3}{4}|x_u-x_v|$. So, let us say let us see the proof . So, we define two notions we say that an index i settles edge $e=\{u,v\}$, if i is the first vertex in the permutation with exactly one of or at least one of with at least one of u and v belongs to $B(s_i,r)$ ok. So, this is the event we call it settling an edge.

Next we say we say that index i cuts edge if exactly one of u and v belongs to $B(s_i,r)$ ok, is cuts an edge. belongs to the multi way cut if and only if. there is an index there is an index i that both settles and cuts  So, that is the only way. So, we call this event of if index i settles edge e that I am calling it $S_i$ and cuts it that event I am calling it $X_i$. So, to prove probability that  there is an index that means, and this events are exhaustive and this events are mutually exclusive.

So, i equal to 1 to k probability that probability $S_i$ and $X_i$ the sum should be less than equal to $\frac{3}{4}|x_u - x_v|$. ok this is what I need to prove. So, first we observe that probability of $X_i$ that it cuts, it cuts only if the radius belongs to the max of you see we have equivalent condition of cutting. So, u belongs to at least exactly one of u and v belongs to the ball. That means, if I look at $1-x_{u,i}$ and $1-x_{v,i}$, r should be sandwiched between minimum and maximum of these two quantities.

This is probability of $X_i$ is exactly same as probability that  minimum of $1-x_{u,i}$ and $1-x_{v,i}$ is greater than equal to r, but less than max $1-x_{u,i}$ and $1-x_{v,i}$. So, if this happens this is the only way exactly one of them belongs to the ball. and this is nothing, but $|x_{u,i}-x_{v,i}|$. So, now, let us see now let us come probability that $S_i$ and $X_i$ is probability that $S_i$and $X_i$ given. So, let l be the index which achieves minimum of $1-x_{u,i}$.

So, suppose l is $argmin_{i \in [k]}\ 1-x_{u,i}$ ok. So, with this so, an index i can settle an index can settle the edge e if or only if it comes before l. So, l occurs  after i times probability l occurs after i. That means, l is the index for which $1-x_{u,i}$ is smallest that means, which is closest in the in probability. in the ball of radius r around $x_{s_i,l}$.

This plus probability $S_i$ and $X_i$ given L occurs before i times probability that l occurs before i. Now, if l occurs before i, i cannot settle. So, in this case this probability is 0 and then this is this and probability of $S_i$ and $X_i$ is less than equal to probability of $X_i$ given l occurs  after i and because we have picked a random permutation probability that l occurs after i is half ok. Now, $X_i$'s are independent that you check that that does not depend on the permutation or random choice of r it. So, $X_i$'s are independent it does not depend on the permutation it depends on the random choice of r.

So, this is same as probability of $X_i$ times half ok. And probability of $X_i$ we have already bound this is equal to $|x_{u,i}-x_{v,i}|$. So, this is $\frac{1}{2}|x_{u,i}-x_{v,i}|$. So, what we get here is $\sum_{i=1}^{k} probability[S_i \wedge X_i]$. So, for l we have to analyze differently this is less than equal to.

because l is the minimum one this is less than equal to $|x_{u,l}-x_{v,l}|$ that is for L, but for other ones this is $\frac{1}{2}\sum_{i\in[k],i\neq l}|x_{u,i}-x_{v,I}|$. So, this is $\frac{1}{2}(x_{u,l}-x_{v,l}+|x_u-x_v|_1)$ . And this we have shown that for any index the difference is at most half times $l_1$ norm. So, this is less than equal to $\frac{1}{4}|x_u-x_v|_1+\frac{1}{2}|x_u-x_v|_1$ which is $\frac{3}{4}|x_u-x_v|_1$ which is exactly what we need to prove. So, we have shown that our algorithm is a $\frac{3}{2}$ factor approximation algorithm ok. So, let us stop here. Thank you.