Approximation Algorithm

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Lecture 37

Lecture 37 : Nonlinear Rounding for MAX-SAT

Welcome. So, in the last class we have seen how running both the algorithms for MAX-SAT two algorithms for MAX-SAT and outputting the better of the two solutions gives us a $\frac{3}{4}$ factor approximation algorithm. So, a natural question is can we design a say $\frac{3}{4}$ factor or even better approximation ratio using only linear programming rounding and the answer is yes and the idea is non-linear rounding. So, let us see this again a very powerful technique it is called non-linear randomized rounding ok. And the idea is you know in the LP rounding randomized LP rounding based algorithm we used a y^* to use it as a probability with which we are setting the variable x_i to be true. So, but we did not need not need to use y^* as raw, we can use any function $f(y^*)$ the only guarantee is that if the function maps from $[0,1]$ to $[0,1]$.

then it perfectly makes sense to set each variable x_i to true with probability $f(y_i^*)$instead of y_i^*. And so, the here the art is how to choose function f so, that we can get a good approximation guarantee and that is what we will see now. So, the idea is instead of setting variable x_i to true probability y_i^*, we set x i to true with probability $f(y_i^*)$ as usual independent of everything else. And, for this what we will see is that any function with certain property gives us a $\frac{3}{4}$ factor approximation algorithm.

So, let f be any function from $[0,1]$ closed interval $[0,1]$ to closed interval $[0,1]$ satisfying $f(x) \leq 4^{x-1}$, but greater than equal to $1-4^{-x}$ for all $x \in [0,1]$. one can ask that does there indeed exist any function where which satisfy this property. Because, we do not need any other property of $f(x)$ this there will exist a function if for all x this inequality that $1-4^{-x}$ is less than equal to 4^{x-1} if this holds. for all x in between 0 and 1 in closed interval $[0,1]$ then there will exist a function, but is it true? Again as usual you can verify that you can you can prove. So, here is a claim it needs a proof that $1-4^{-x}$ is less than equal to 4^{x-1} for all $x \in [0,1]$.

In particular, there exists a function f that satisfies the above inequalities. So, again for this kind of inequalities it is often easier to verify by drawing plots of the function. So, if I plot the function here x here y 0 1 0 1. So, if you plot 4^{x-1}. So, at x equal to 1 this function is 1 and at x equal to 0 this function is $\frac{1}{4}$.

So, here if it is 0.5. So, 4^{x-1} looks like this. This is 4^{x-1}. On the other hand $1-4^{-x}$ at x equal to 0 it is 0.

So, it starts here and at x equal to 1 it is $\frac{3}{4}$. So, it is end somewhere here. So, this is how the function look like and they do not cross at any point. So, these two are same at x equal to maybe 0.

5. So, what is the algorithm? So, let me write it as a theorem let (y^*, z^*) be an optimal of the LP relaxation of MAX-SAT, the algorithm which sets each variable x_i to true with probability $f(y_i^*)$ independent of everything else achieves an approximation factor of at least $\frac{3}{4}$. So, instead of setting each variable to true with probability y^* we set it to true with probability $f(y_i^*)$ ok. So, here again as usual we need to bound the probability with which a clause C_j is satisfied. probability that $C_j, j \in [m]$ is satisfied this is 1 minus the probability that it is not satisfied and the only way it is not satisfied is that all the variables which appear positively is said to false and all the variables which appear negatively is said to true. So, this is $1-\prod \square$ the probability that x_i is said to false is $1-f(y_i^*)$ times for all $i \in N_j$ the probability that x_i is said to true which is $f(y_i^*)$ ok.

Now, what is the property of function? The function is $f(x)$ is in between $1-4^{-x}$ and less and 4^{x-1}. $1-4^{-x}$ is less than equal to $f(x)$ less than equal to 4^{x-1}. So, it gives us 2 bounds we want to show this. So, we need lower bounds for $f(y_i^*)$ and $1-f(y_i^*)$. So, from here we get $f(x)$ is less than equal to 4^{x-1} and $1-f(x)$ is less than equal to 4^{-x}.

So, using these 2 bounds what we can write here is this is $1-\prod 4^{-y_i^*}$ times $\prod 4^{-y_i^*+1}$. So, this is $1-4^{\sum -y_i^* + \sum -y_i^* + 1}$. So, if you take again I want to use that I want to write this in terms of z_j's and recall in the linear program we had a constraint that for all $j \in [m]$ $\sum y_i^* + \sum 1-y_i^*$. this is greater than equal to z_j.

The idea being z_j can be set to 1 only if this sum is at least 1. If the sum is less than 1. z_j cannot be set to 1. So, if I take this minus outside this is $1-4^{\sum -y_i^* + \sum -y_i^* + 1}$. Now, this is greater than equal to z_j and this is the inequality direction we need because of 2 minus signs. this is greater than equal to $1-4^{-z_j^*}$ ok.

And again, you look at the function. I want to write this down; I want to bring z_j^* down from the exponent as a product of some constant times z_j. To do that, we will again use the notion of concavity: $g(z)=1-4^{-z}$ is concave. On the interval (0, 1), g(0) is 1, and g(1) is also 1. If I plot a line, I can choose any value of z.

This is z; this is g. z is less than or equal to g, and g is greater than or equal to the green part. The sentence So, what do we have? is indeed correct; no changes are needed. So, what is g(0)?

So, this is 1 g 1, which equals $1-\frac{1}{4}$, resulting in $\frac{3}{4}$. So, this is $\frac{3}{4}$. Sorry, g(0) is 0, not 1. It is $1-4^0$, which equals 1. Is that okay? So, g(z) is greater than or equal to 0; additionally, it is $g(0)+z\,g(1)$, which equals $0+\frac{3}{4}z$.

Therefore, this is greater than or equal to z_j^*. Since z_j^* belongs to the closed interval $[0,1]$, this holds for all z in the closed interval $[0,1]$. ALG is the $\sum w_j$. Therefore, the expectation of ALG can be expressed as the $\sum w_j$ multiplied by the probability that z_j satisfies C_j for $j\in[m]$. This is greater than or equal to $\frac{3}{4}$ because each of these probabilities is greater than or equal to $\frac{3}{4}z_j^*$.

So, this is greater than or equal to the $\sum \frac{3}{4} w_j z_j^*$, and the $\sum w_j z_j$ is the objective function of this LP relaxation, while y^* and z^* are the optimal values. So, this summation represents the optimum of the linear programming problem.

The approximation factor of our algorithm is at least $\frac{3}{4}$.

A natural question is whether we can obtain an approximation factor for this problem that is better than 3/4. The original sentence is indeed grammatically correct. There is no need for any corrections. We partially answer this in the sense that if we use that linear programming relaxation, we cannot have an algorithm better than 3/4. The original sentence is indeed grammatically correct, so no changes are necessary.

However, if you would like a slightly more concise version, it could be rephrased as: "If we use the $\sum w_j z_j$ for the LP relaxation to bound the optimal value, we find that $opt \leq \sum w_j z_j$ since it is a maximization problem." This version removes the comma before "since," which is acceptable in this context. Then there is no better approximation algorithm than a factor of 3/4. The sentence is indeed grammatically correct; no changes are necessary. The sentence is already correct as it is: "What is the integrality gap?" The integrality gap occurs when we have an integer linear programming (ILP) problem.

So, it is the maximum of all ILPs for I, which means the best integral solution you can obtain from the LP optimum of I. This applies to a maximization problem; for a minimization problem, you should consider how to define it in a very natural way. Now, this is the same as saying that ILP optimization is equivalent to optimization.The reason for this is that if you look at an instance of the lower bound, we need to show that it is greater than or equal to $\frac{3}{4}$

I just need to provide an instance where this value is at least $\frac{3}{4}$. So, consider this instance: x_1 or x_2, $\bar{x_1}$ or x_2, x_1 or $\bar{x_2}$, and $\bar{x_1}$ or x_2. This means that all four possible clauses for using these two variables are included. Obviously, the optimal value, which is equal to the ILP optimal value, is 3.

No matter how you assign x_1 and x_2, exactly one of these four clauses will remain unsatisfied, while the other three clauses will be satisfied. On the other hand, if you write down the LP, take it as homework to write down the LP relaxation and show that the optimal value of the LP is equal to 4. Hence, the maximum of $\frac{opt(I)}{LP-opt(I)}$ is greater than or equal to $\frac{3}{4}$. Thank you!