Approximation Algorithm

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Lecture 36

Lecture 36 : Best of Two Solutions for MAX-SAT

Welcome. So, in the last couple of lectures we have seen few randomized algorithms for MAX-SAT problem. In this lecture we will again you see very powerful technique that often if we have more than one algorithm for a problem, then running all the algorithms and outputting the best ah um solution is a good idea. So, the technique let us call it choosing better of many solutions . So, the technique choosing better of best of many solutions output by different algorithms. So, our problem is MAX-SAT and the first algorithm that we have seen simply sets each variable to true and false with equal probability independent of everything else.

So, I will go set each variable to true or false with equal probability independent of everything else. So, if I have a clause C_j with l_j many literals, then that clause is satisfied by this algorithm with probability $1-2^{-l_j}$. A clause C_j with l_j literals is satisfied by this algorithm with probability $1-2^{-l_j}$. This is so, because there is exactly one assignment for each literal.

and in particular there are 2 to the power 1 of 2^{l_j} many assignments of this l_j literals 1 assignment makes the clause C_j not satisfied. Every other assignment out of 2 to the power l_j assignments of this l_j variables satisfies this clause. And the second algorithm So, let us take the randomized rounding based algorithm which first writes down the integer linear programming formulation. and then relaxes it to linear program LP formulation and then uses the variables y_i's to set the variables in the formula true with probability y_j ok. We start with the optimal solution of the linear program.

So, that is our algorithm 2. So, what is algorithm 2? Solve the relaxed LP formulation of MAX-SAT. Let (y^*, z^*) be an optimal solution Boolean variable x_i to true with probability y_i^* independent of everything else ok. Here we saw that a clause C_j with l_j

literals is satisfied with probability $1-\left(1-\frac{1}{l_j}\right)^{l_j} z_j^*$ with probability greater than equal to this ok. So, what is our algorithm? Our algorithm is run these two randomized algorithms, each algorithm gives me a solution, we check which algorithm satisfies which algorithm which assignment is better that means, which algorithm has more sum of weight of satisfied clauses and an output the algorithm.

we run both the algorithms. $f_1:\{x_1,\ldots,x_n\}\rightarrow\{true, false\}$ and $f_2:\{x_1,\ldots,x_n\}\rightarrow\{true, false\}$ be the assignments output by the algorithms. we check which between f_1 and f_2 maximizes the weighted sum of satisfied clauses and output it ok.

So, for this approach to work it should be possible to evaluate the goodness of a solution because this is required. Next we show that the performance guarantee the approximation factor is of this algorithm is better than the approximation factor of both ALGO-1 and ALGO-2 and which is surprising and which is often the case of this kind of algorithms and the intuitive idea is that ALGO-1 works better if the clauses are large. For example, if l_j is large then this probability is close to 1. So, ALGO-1 works better if l_j is large. if l_j is large with close to 1 probability the clause is satisfied.

On the other hand if l_j's are small for example, if l_j is say 1 then it is 1. So, just z_j^* and $\sum z_j^*$ is LP opt remember. So, this ALGO-2 is better if l_j is small. So, the instances where the output where the algorithm each of this algorithm works works better are disjoint and complementary that is important. So, the this is the intuition the region of inputs where l go 1 is ah works better is complement this is the idea not the precise mathematical statement complement to the complement of the region of inputs where algo 2 is better.

Hence, not surprising if on any instance if I run the run both the algorithms and output the better solution, then for all inputs the overall output of this algorithm this master algorithm which runs both the algorithms and outputs the better solution has better approximation guarantee and this is what we show in the next theorem. our algorithm has an approximation factor of at least $\frac{3}{4}$ proof. So, let W_1 be the random variable denoting the value of the solution. output by ALGO-1 and W_1 and W_2. ALGO-1 and ALGO-2 So, what is the ALG then? ALG is the value of the solution output by this master algorithm.

So, ALG is by definition max of W_1 and W_2 and to show I want to show that expectation of alg which is expectation max of W_1 and W_2 is greater than equal to $\frac{3}{4}$ times opt that is what I want to show. So, to obtain that we need to use the inequality that max of W_1 and W_2 is greater than equal to the average. because expectation of max it is not need, but if it is sum we know that we can apply or use linearity of expectation. So, expectation of max taking expectation on both side max of W_1 and W_2 this is greater than equal to $E\left[\frac{W_1}{2}+\frac{W_2}{2}\right]$. Applying linearity of expectation this is $\frac{1}{2}E[W_1]+\frac{1}{2}E[W_2]$.

ok and we know that expectation of W_1 it satisfies each clause with this much probability $1-2^{-l_j}$ and so, the expectation of W_1 is $\sum_{j=1}^{m} w_j(1-2^{-l_j})$, here half is there $\frac{+1}{2}\sum_{j=1}^{m}(1-(1-\frac{1}{l_j})^{l_j})w_j z_j^*$ ok. Now, I want to write it together. So, this is recall z_j^* lies in between 0 and 1. So, this is greater than equal to there is a times 1.

So, 1 is greater than equal to z_j^*. So, if I replace this 1 with z_j^* I get a lower bound. $\sum_{j=1}^{m} w_j z_j^*(1-2^{-l_j})$. So, let us put half also inside. sorry half inside plus $\frac{+1}{2}\sum_{j=1}^{m}(1(1-\frac{1}{l_j})^{l_j})w_j z_j^*$ ok.

So, now what we will show that. claim that this term is greater than equal to $\frac{3}{4}$ for all integer $l_j \geq 1$. So, we will prove this shortly, but assuming this then we what we have is this is greater than equal to $\frac{3}{4}\sum_{j=1}^{m} w_j z_j^*$ and this is nothing, but LP of the relaxed LP. So, this is equal to $\frac{3}{4}LP-opt$ which is greater than equal to $\frac{3}{4}opt$ ok. So, all we need to show is this claim that this term is greater than equal to $\frac{3}{4}$ for all integer $l_j \geq 1$.

So, let us prove that claim. for every integer $l \geq 1$ average of $(1-2^{-1})$ and $(1-(1-\frac{1}{l})^{l})$ this is greater than equal to $\frac{3}{4}$. So, for $l=1$. what is this? This is $\frac{1}{4}$ and this is $\frac{1}{2}$.

So, this is $\frac{3}{4}$. So, for $l=1$ this is true for $l=2$ let us see what it is $\frac{1}{2}(1-\frac{1}{4})+\frac{1}{2}(1-(1-\frac{1}{2})^{2})$ which is $\frac{3}{4}$. So, for $l=1$ and 2 it holds now we will prove it for

$l \geq 3$. for L greater than equal to 3, what we have is you can show that $\left(1-\left(1-\frac{1}{l}\right)^l\right)$ this is greater than equal to $1-\frac{1}{e}$.

ok. The idea is as l tends to infinity this tends to as l tends to infinity this $\left(1-\frac{1}{l}\right)^l$ to $\frac{1}{e}$ from bottom from left side. So, this is at least $1-\frac{1}{e}$ when $l \geq 3$, this you can verify and prove also. And $\left(1-2^{-1}\right)$ Now, for $l=3$ this is $\frac{7}{8}$ and $1-\frac{1}{8}$ it is $\frac{7}{8}$. So, if l is more then 2^{-l} is even smaller than $\frac{1}{8}$. So, this is greater than equal to $\frac{7}{8}$ ok.

So, $\frac{1}{2}\left(1-\left(1-\frac{1}{l}\right)^l\right)+\frac{1}{2}\left(1-2^{-1}\right)$ this is greater than equal to this is for $l \geq 3$ $\frac{1}{2}\left(1-\frac{1}{e}\right)+\frac{1}{2}\frac{7}{8}$ which you can verify this is this is roughly 0.753 which is greater than equal to $\frac{3}{4}$. So, this proves the claim, but how one guess this thing what is the right constant. So, for that what it is often useful to plot these graphs.

So, if you simply plot these graphs ok. So, this here l is there and that probabilities can be at most 1. So, maybe let us call it and it is at least 0.

5. So, let us start with 0.5 this and here it is 1 suppose it is 2, 3, 4 and so on and then if you plot the first function what is $1-2^{-l}$. Now, for $l=1$ this is $\frac{1}{2}$. So, this function is like is like this this this how it grows this is this is $1-2^{-l}$ another function is $\left(1-\left(1-\frac{1}{l}\right)^l\right)$ ok. So, for $l=1$. So, suppose this is 1 and this function grows like and so on.

So, and if you look at the maximum this is the maximum of these two functions which is always greater than $\frac{3}{4}$. but that is not enough what we need to show is the average of these 2 functions is greater than $\frac{3}{4}$. So, if you plot the average it looks like this. So, the average of these 2 functions from 2 onwards it is greater than equal to 4 between 1 and 2 average drops below $\frac{3}{4}$, but at 1 at l equal to 1 it is at $\frac{3}{4}$ and because l is only integral l can take only integer values and that is what we need we can show we are able to show that the average of these two functions is at least $\frac{3}{4}$ for any integer value of l greater than equal to 1 ok.

So, this is how you first get the right constant and then once you get the right constant you can prove it ok. So, let us stop here. Thank you.