**Approximation Algorithm**

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**Week – 01**

**Lecture 03**

Lecture 03 : Overview of LP Duality and Complementary Slackness

Welcome, in the last lecture we have seen deterministic rounding of linear programming relaxation for set cover and we obtained a F factor approximation algorithm. So, today we will see another technique which is called rounding a dual programming. So, for the same set cover problem we can write the linear programming relaxation as follows also. So, consider the set cover problem and we can think of we are charging each element of the universe some price say $y_i$ for covering it. So, we charge $y_i$ . for the element $e_i$ of the universe to get it covered. ok. And our goal is to cover all elements with total with minimum total cost. and what is the total cost? It is sum of the costs of all elements of the universe $\sum_{i=1}^{n} y_i$ ok. And what are the constraints? if I pick a set each set has a cost each set $S_j$ has a cost $w_j$ and it can cover all elements in its set with total cost at least W j.

So, for every element for every set not element we can cover all the elements of $S_j$ with total cost $w_j$ . Hence, we have for every set if I sum over the elements $y_i$ this cost must be greater than equal to $w_j$ , this is for all $j \in [m]$ . We have m sets it says that if I charge each element $e_i$ to $y_i$ , then all the elements in a set $S_j$ can pay for themselves for getting it covered, if summation of the charge of the costs of the elements is at least the weight of that set. So, what is the linear program we have? the final linear program relaxation for the set cover. is the following.

It is minimize $\sum_{i=1}^{n} y_i$ there are n elements in the universe subject to for every set j in the collection summation of the elements of the set their charges $y_i$ must be greater than equal to $w_j$ . ok and $y_i$ 's are greater than equal to 0 non negative they can cannot be negative this is for all $i \in [n]$ . Now, we use this linear program to obtain a to obtain a approximation algorithm for set cover. It turns out that this linear program is the dual linear program of the earlier linear programming relaxation that we have we had written for the set cover. This is the dual of the linear program relaxation of the set cover. What is the linear programming relaxation for the set cover problem we had seen in the last class let us recall. Our goal is to minimize the sum of the weights of the sets we pick and for every set $S_j$ we have a variable called $x_j$ and our goal is to minimize that set. So, minimize $\sum_{j=1}^{m} w_j x_j$ and the condition was that for every element $i \in [n]$ for every element in the universe look at the sets which contains that element. you must pick at least one such set ok. And in the exact formulation we had $x_j \in \{0,1\}$ this was the ILP formulation $ILP-opt=opt$ of the set cover and we relax this to linear programming constraint that $0 \leq x_j \leq 1$ , this is for all $j \in [m]$ . and then we observe that because assigning any value to $x_j > 1$ can never lead to optimal solution because we are minimizing $\sum_{j=1}^{m} w_j x_j$ and the constraint is $\sum x_j \geq 1$ . We can also get rid of this constraint this part of the constraint and So, you can get rid of this part of the constraint and the final linear program that we have is $x_j$ is greater than equal to 0. Now, this linear program when we have we write dual sometimes the original linear program is called refer to as primal linear program. So, this one we call primal linear program. Now if you do not know what is Primal Linear Program, how to get dual, it is important that you learn it yourself that we assume as a prerequisite.

There are plenty of materials which available in the internet and which covers this Primal Linear Program, Dual Linear Program how to obtain the dual linear program of a given linear program and that is a fairly mechanical procedure. And if you apply that procedure to this primer linear program, then you will obtain the this linear program which is the dual linear program. So, this we call the dual The approximation algorithm book by Williamson and Chamois also covers this primal and dual linear programs and their relationship in the appendix you can study from there also. Now, what is the relationship between primal linear program and dual linear program? So, it turns out there is something called weak duality law which let us prove in this weak duality theorem. It says that. Let us write that in with respect to this primal linear program and that dual linear program. So, let $y_i, i \in [n]$ be a feasible solution of the dual LP. what is the value of the objective function of the dual LP at this $y_i$ 's, this is the $\sum_{i=1}^{n} y_i$ Then $\sum_{i=1}^{n} y_i$ to this is less than equal to this forms a lower bound of primal-opt ok. So, any dual feasible solution what is dual feasible solution? Any assignment to the variables $y_i$ 's which satisfy this constraints that is a dual feasible solution for at that dual feasible solution the value of the dual objective function is a lower bound on the primal optimal.

So, let us prove it. and this is true this holds for arbitrary primal and dual corresponding dual linear programs, but again let us prove this only for with respect to this primal and dual linear program. So, let $x_j$ primal opt if it exists. So, in this case it exists. So, let we do not need to write that ok.

So, taking this primal linear program is feasible in this case for example, if we set all $x_j$ to be 1 that is a feasible solution. So, it take any feasible primal solution. So, let $x_j, j \in [m]$ be any primal feasible solution. Now, what we will show is that $\sum_{i=1}^{n} y_i$ is less than equal to the value of the primal objective at this primal feasible solution. So, let us write $\sum_{i=1}^{n} y_i$ this you can write as $\sum_{i=1}^{n} y_i \times 1 \leq \sum_{i=1}^{n} y_i \sum_{j \in [m] : e_i \in S_j} x_j$ . Now, whenever we have double sum it is it is it has often been seen that exchanging the double sum often gives useful insights.

So, let us write it exchange the double sums. So, we write j first. $\sum_{j=1}^{m} x_j \sum_{e_i \in S_j} y_i$

And now we use the dual constraint this constraint that for every set $S_j$ the sum of the $y_i$ values of the elements in that set is greater than equal to $w_j$ . So, this is for the dual linear program it should be that this should be less than equal to $\sum_{j=1}^{m} w_j$ whenever we have a primal program which is a minimization problem the dual must be maximization problem. And the idea is we want to maximize the sum of the weights sum of the charges, but for each set $S_j$ this is less than equal to. that total amount we should charge is at most $w_j$ , it is like by paying $w_j$ we can cover all elements in that set $S_j$ . So, and this we want to maximize.

So, now, we use this dual linear program this is less than equal to $\sum_{j=1}^{m} w_j x_j$ . So, in particular whenever you have a primer linear program which is a minimization problem the dual should be a maximization problem and vice versa ok. And this is the this is the primal objective, primal optimization function value. is the primal of optimization function value at $x_j, j \in [m]$ . Now, because primal is a minimization problem this is less than equal to primal of So, we have shown that that dual any dual feasible solution if I have that dual objective function is less than equal to primal opt.

So, this is the weak duality theorem. Now, there is another very powerful result which is called strong duality theorem. it says that if both primal and dual are feasible solutions if both primal and dual linear programs are feasible, then primal opt equal to dual opt. In the context of set covered clearly primal is feasible we can set all $x_j$ to be 1 and of course, the dual is also feasible we can set all $y_i$ 's to be 0 because $w_j$ 's are greater than equal to 0. So, we see that for the primal dual pair of linear programs of the weighted set cover problem, primal opt equal to dual opt.

Now, you see for primal opt to be dual opt. So, if I take 2. So, let $x_j^*, j \in [m]$ be a primal optimal solution. and why I star be a dual optimal solution. this inequality, but instead of $y_i$ we will write $y_i^*$ instead of $x_j$ we will write $x_j^*$ .

$\sum_{i=1}^{n} y_i^*$ this is less than equal to this this is less than equal to $\sum_{i=1}^{n} y_i^* \sum_{j \in [m]: e_i \in S_j} x_j^*$ this is less than equal to this one. So, this line we have written now we are writing this line this is less than equal to or this is equal to let us write this line also $\sum_{j=1}^{m} x_j^* \sum_{e_i \in S_j} y_i^*$ ok. This is less than equal to $\sum_{j=1}^{m} w_j x_j^*$ . Now this is dual opt and this is primal-opt. Now for set cover we have seen that it strong duality theorem says that dual opt equal to primal opt that means, these two inequalities must be equality. This is called complementary slackness condition. So, what is complementary slackness conditions It says that if $x^*$ and ok. So, what are the conditions? So, let $(x_j^*)_{j \in [m]}$ and $(y_i^*)_{i \in [n]}$ be primal and dual solutions. whenever. So, the conditions are first condition is if $x_j^*$ greater than 0, then the corresponding dual constraint is tight.

So, for each primal variable we have a corresponding dual variable for all $j \in [m]$ and same the other way. This ensures that this first inequality sorry this ensures that the second inequality holds with equality tight means an equality constraint or equality or constraint is tight means it holds with equality. In the other way if $y_i^* > 0$ , then the corresponding primal constraint is tight for all $i \in [n]$ . So, second condition ensures that the first inequality this one holds with equality. See both conditions hold both inequalities hold with equality and that means, both $(x_j^*)_{j \in [m]}$ and $(y_i^*)_{i \in [n]}$ are optimal solutions.

So, it says that $(x_j^*)_{j \in [m]}$ and $(y_i^*)_{i \in [n]}$ are primal and dual optimal solutions if 1 and if and only if 1 and 2 hold ok. So, let us stop here we will see how this complementary slackness conditions we use it very crucially in algorithm design and analysis of approximation algorithms ok. Thank you.