Artificial Intelligence for Economics

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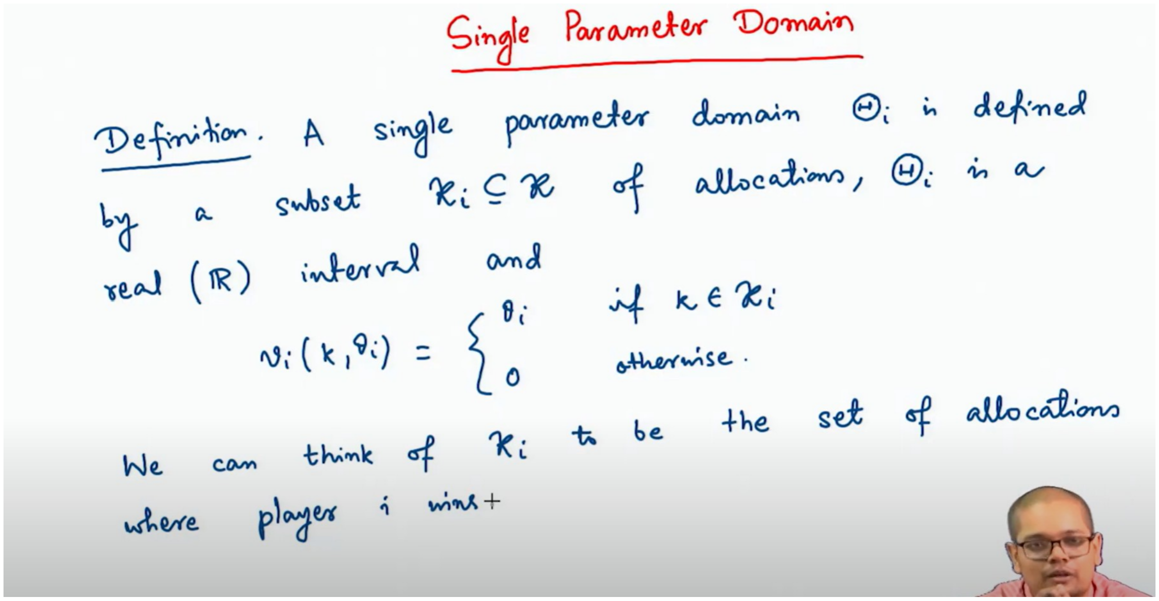
Week – 07

Lecture - 32

Lecture 32 : Single Parameter Domain and Myerson Lemma

Welcome in the last lecture we have studied VCG auction and these are celebrated result in mechanism design and auction theory. So, today we will see another interesting result which is called Myerson's lemma. So, it is called single parameter domain and in this domain we have Myerson's lemma. So, what is single parameter domain? So, often the typesets has a special structure for example, it may not be any arbitrary set till now we have assumed that the typeset Θ_i is an arbitrary set, but often it is a say for an auction scenario type encodes the valuation of the player which is a real number. Now, single parameter domain intuitively speaking is a domain where the type set or a type value can be can be represented by one say one real number ok. So, formally what is the definition of single parameter domain? again this is not of not a full fledged definition, but this definition is good enough for meeting our purpose.

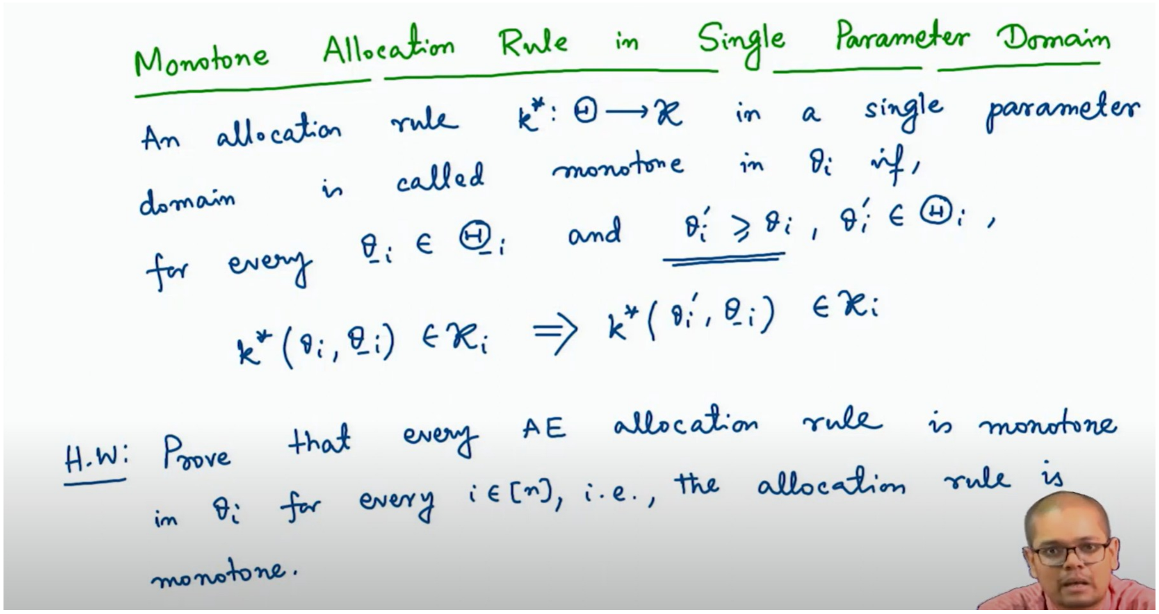
What is single parameter domain? A single parameter domain is defined by a subset $K_i \subseteq K$ of allocations Θ_i is a real interval or real interval that means, an interval on real line and v_i of write here this way v_i of an allocation with type θ_i has two options. If it is θ_i if $k \in K_i$ that means, this particular set and 0 otherwise. So, we can think of K_i as the set of allocations where player i wins and in other allocations player i loses. So, those allocations player i values at 0.



So, we can think of K_i to be the set of allocations where player i wins. So, in general when theta i could be arbitrary not single parameter domain, we have seen that any allocatively efficient allocation rules can be coupled with a suitable payment scheme. So, that the mechanism is dominant strategy incentive compatible. We loosely say that allocation monotone allocation rules are implementable using groves mechanism using groves payment scheme. So, here in single parameter domain we will see that every there is something called monotone allocation rule and every monotone allocation rule is implementable in single parameter domain.

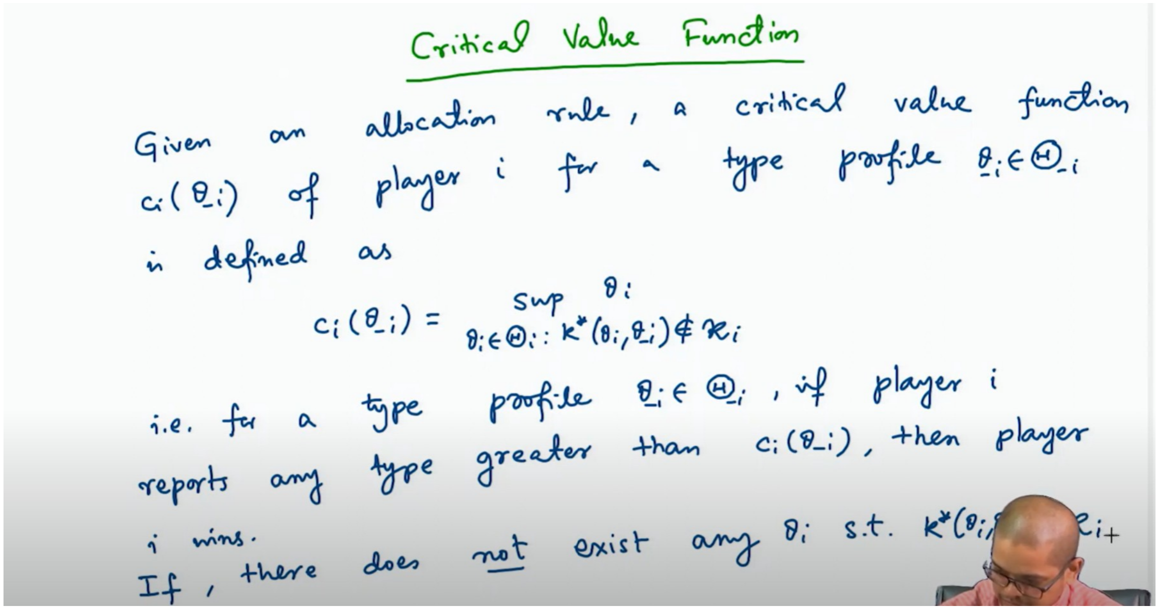
So, what is monotone allocation rule? it makes sense only in single parameter domain monotone allocation rule in single parameter what is it? An allocation rule k^* from $\Theta \to K$ in a single parameter domain is called monotone is called monotone in Θ_i recall k^* the input is the tuple $(\theta_1, \theta_2, \ldots, \theta_n)$. If for every $\theta_{-i} \in \Theta_{-i}$ in for every type profile of other players and $\theta_i' \geq \theta_i$, $\theta_i' \in \Theta_i$. So, and every type of player i which is greater than θ_i which is θ_i' $k^*(\theta_i, \theta_{-i}) \in K_i$ if it happens that in the type profile (θ_i, θ_{-i}) player i wins that means, $k^*(\theta_i, \theta_{-i}) \in K_i$ wins.

player i will continue to win if its type increases. Here you see we are crucially assuming that using the fact that Θ_i is an interval in real in real line. Otherwise for arbitrary abstract objects we cannot write greater than equal to. So, this is $k^*(\theta_i', \theta_{-i}) \in K$ ok. Increasing the valuation of a particular player keeping every others valuations fixed cannot result in losing if that particular player was winning before.



So, that is called monotone allocation rule ok. And Myerson's lemma says that every monotone allocation rule is implementable. But before that we have said that it is single parameter domain is a special case of general domain where we have VCG mechanism. So, it makes sense it should happen that whichever payment whichever allocation rule are implementable in arbitrary setting they should continue to be implementable in implementable in single parameter domain and in single parameter domain monotone allocation rules are the only implementable allocation rules. So, it should be the case that every allocatively efficient allocation rule is monotone, but that needs a proof I will leave that proof as homework it is an easy proof.

So, homework prove that every allocatively efficient allocation rule is monotone monotone in θ_i. for every $i \in [n]$ ok. So, a monotone allocation rule is a allocation rule which is monotone in every θ_i that is the allocation rule is monotone. ok. The allocation rule is monotone, but there exist monotone allocation rules which are not allocatively efficient.



So, in single parameter domain we can implement all allocatively efficient rules and something more. So, the second homework is prove that there exists a monotone allocation rule. monotone allocation rule in a single parameter domain which is not allocatively efficient ok. So, that means, monotone allocation rule is a strict superset of allocatively efficient allocation rules. So, these are monotone allocations rules ok and here we have allocatively efficient allocation rules ok good.

So, for monotone allocation rules there is something called critical value function. So, if we go in the next page there is something called critical value function. What is critical value function? So, given an allocation rule. a critical value function c_i of it is a function of the type profile of other players $c_i(\theta_{-i})$ of player i. So, every player has a critical value function for a type profile $\theta_{-i} \in \Theta_{-i}$ is defined as $c_i(\theta_{-i})$ is supremum of all θ_i where player i loses.

So, this is supremum over $\theta_i \in \Theta_i$ such that $k^*(\theta_i, \theta_{-i}) \notin K_i$. That means, for the type profile θ_{-i} you look at the set of all types of player i where player i loses that means, $k^*(\theta_i, \theta_{-i}) \notin K_i$ you take the supremum of that. So, for a type profile θ_{-i} if and the allocation rule is monotone no not allocation rule is monotone that is not required if the type profile is θ_{-i} type profile of other players. Then, if player i bids anything greater than $c_i(\theta_{-i})$ then player i wins ok. So, that is for type profile $\theta_{-i} \in \Theta_{-i}$.

If player i reports any type greater than $c_i(\theta_{-i})$, then player i wins ok. And so, if this set is not defined so, if if there does not exist any θ_i such that $k^*(\theta_i, \theta_{-i}) \notin K_i$ at that θ_{-i} c_i is undefined ok. So, here is a beautiful characterization of DSIC social choice functions in single parameter domain. which again we will not prove although the proof is not very difficult, but let us let us not go into the proof that is not the intention of this course.

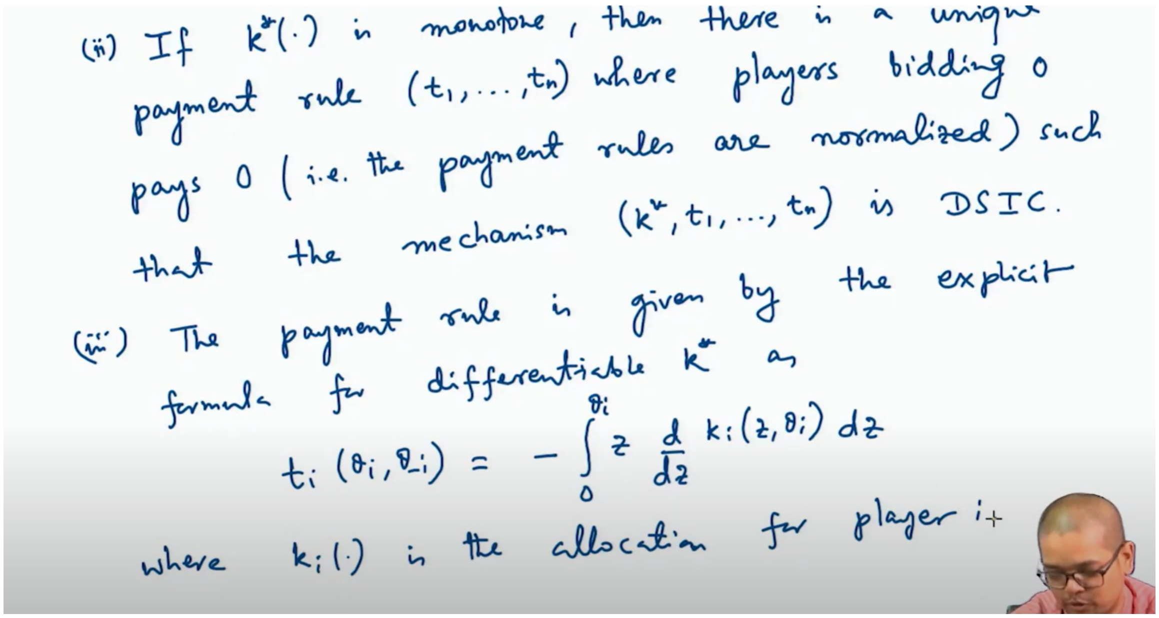
So, characterization of DSIC social choice functions in single parameter domain. A social choice function f recall in a single in a single parameter domain which is a quasi linear setting it has two parts one is allocation function another is payment. Social choice function in a single parameter domain is dominant strategy incentive compatible and losers do not pay Again losers make sense only because it is a single parameter domain for whom the allocation function does allocation does not belong to K_i they are the losers. In arbitrary quasi linear setting loser does not make sense. So, losers do not pay anything.

if and only if it is a characterization the following holds. The first condition is the allocation rule k^* is monotone in every θ_i, every winning player essentially pays her critical bid. every winning player essentially pays her critical bid that is critical value. that is $t_i(\theta_i, \theta_{-i}) = -c_i(\theta_{-i})$ if $k^*(\theta_i, \theta_{-i}) \in K_i$. So, this is the payment and 0 otherwise it pays that is why it is negative $-c_i$ and 0 otherwise.

So, this is not very difficult to prove, but again let us leave the proof. So, this is the characterization of implementable social choice functions in single parameter domain. Again we see basically that every monotone allocation rules can be clubbed with the payment rule which makes the social choice function implementable in dominant strategy equilibrium, its dominant strategy incentive compatible. But recall as we said that you know these our definition of. single parameter domain is bit simple we have assumed that the allocations can be divided into two parts one is winning another is losing.

So, it may not always be such all we all we the most important thing all we care is that theta i is a real interval or it can be encoded as one single parameter. that is where the name single parameter domain. So, in those are more general setting we have Myerson's lemma. So, what is Myerson's lemma? it says that we have the following in the single parameter domain. what do we have? An allocation rule $k^*: \Theta \rightarrow K$ is implementable by having suitable payment structures in dominant strategy equilibrium if and only if it is monotone ok.

Second one if k is monotone, then there is a unique payment scheme is unique payment rule $t_1, \ldots, t_n$ where unique subject to this condition where players bidding So, these payments payment rules are called normalized payment rules that is the payment rules are normalized. then there is a unique payment rule $t_1, \ldots, t_n$ where buyers bidding 0 pays 0 such that the social choice function or mechanism $(k^*, t_1, \ldots, t_n)$. is dominant strategy incentive compatible ok. The third one gives a payment formula. The payment rule is given by the explicit formula for differentiable k this function is differentiable as what is the formula? Formula is $t_i(\theta_i, \theta_{-i}) = -\int_0^{\theta_i} z \frac{dk_i(z, \theta_i)}{dz}$.



So, it basically So, where what is k? Where k_i is the allocation for player i. So, this is the allocation of player i if the payment allocation function can be can be distributed like $(k_1, k_2, \ldots, k_n)$ and then we have and there that is differentiable then we have the formula like this ok. So, in the next class we will see and concrete a use case of this Myerson's lemma and Myerson's payment scheme in a very important application ok. Let us stop here. Thank you.