Artificial Intelligence for Economics

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Week – 06

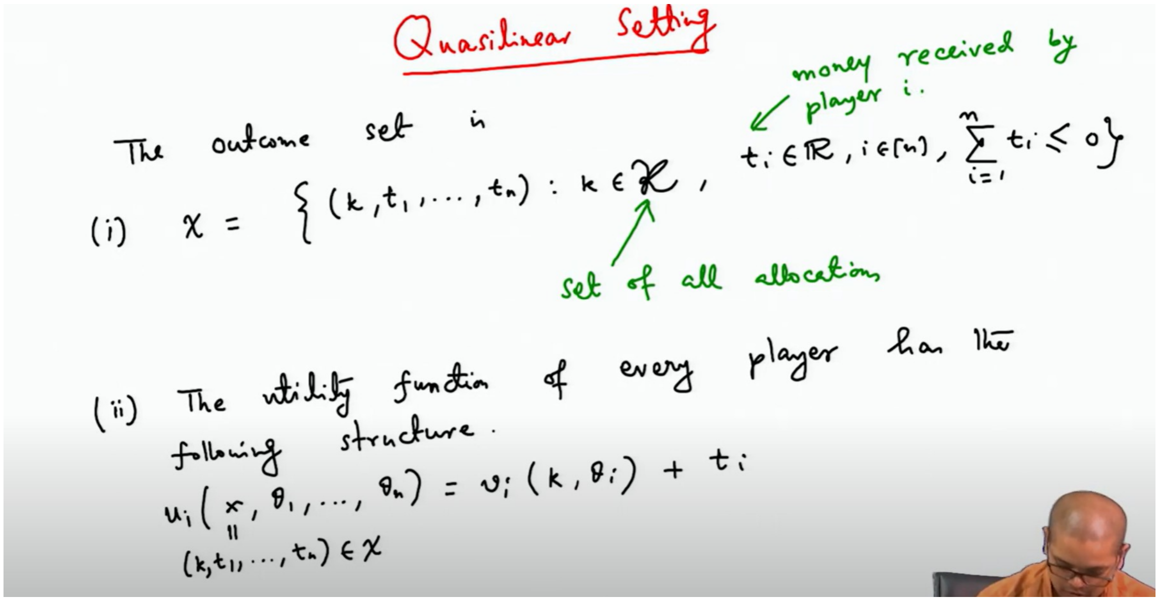
Lecture - 30

Lecture 30 : Groves Mechanism

In the last lecture, I have given you a broad overview of mechanism design that will give you a feel of mechanism design. We have discussed what are social choice functions and what we mean by direct mechanisms, what do you mean by implementing a social choice functions. There are two kinds of implementations. one is dominant strategy in equilibrium implementation in dominant strategy equilibrium or DSE. The such social choice functions are called DSIC social choice functions dominant strategy incentive compatible and the other is implementation in Bayesian Nash equilibrium which is called BIC Bayesian Nash incentive compatible social choice functions. And, then we we at the end we have seen the Gibbard-Satterwaite theorem which says that under very mild assumptions under very desirable conditions only dictatorship social choice functions are implementable.

So, today we will study what is called a quasi linear setting. it assume it assumes extra structure on the outcomes recall in in a social choice function. The outcome set x could be arbitrary that we are not allowing here in quasi linear setting and hence thereby, thereby we are bypassing the negative result of Gibbard-Satterwaite theorem. So, the main assumption is the how the outcome set is X equal to it must look like a tuple or goal is to study the auction kind of scenario.

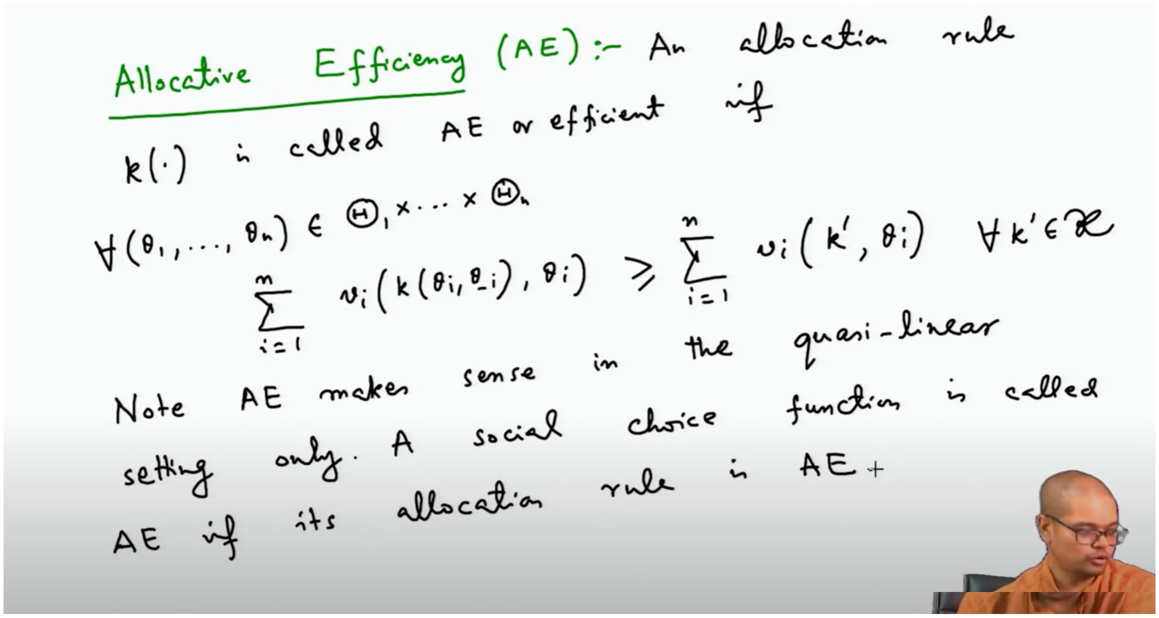
So, it must look like a tuple the first one will be an allocation in the second and the next ones will be if there are n players this will be payments ok, where small k is in cal K where cal K is the set of all allocations. So, there is some object which we want to allocate. So, this is the set of all allocations. ok and then we have this t_i's $t_i \in R$ these are real numbers $i \in [n]$ and $\sum t_i \leq 0$. So, think of t_i as the amount of money received by player i.



So, $\sum t_i \leq 0$ means the mechanism does not need outside money any any monetary support from outside ok. So, this is very important assumption that the outcome set looks like this that is one. The second one is you see outcome set is now infinite. The second one is utility function the utility function of every player has the following structure. what is it u say $u_i(x, \theta_1, \ldots, \theta_n)$ it has 2 parts.

Now, x has itself is itself looks like a tuple. So, it is u_i of suppose x equal to this tuple k $(t_1, \ldots, t_n)$ is in X. Now, this is then it looks like there is a some valuation every player has some valuation function v_i which depends only on allocation and its type θ_i plus payment. because t_i is the amount of money received by player i. So, we write here ok.

So, this is called quasi linear setting and in this setting there exist many social choice functions which are dominant strategy incentive compatible. So, here we state and prove what is called Grove's theorem ok what is groves theorem. So, let so, a social choice function f which takes the type profile as inputs $(\theta_1, \theta_2, \ldots, \theta_n)$ and it outputs an allocation function and payments ok. So, let f be an allocatively efficient social choice function.



will soon see what is allocatively efficient social choice function mean. Then f is dominant strategy. incentive compatible DSIC in short. If the payment functions satisfy the following structure that $t_i(\theta_i, \theta_{-i})$ this is a convenient shorthand which is same as $(\theta_1, \theta_2, \ldots, \theta_n)$ but this focuses on θ_i this is the sum of valuations of all players except i ok. So, $v_j(k, \theta_i, \theta_{-i})$ that sum of and v_j takes allocation and the type of player j.

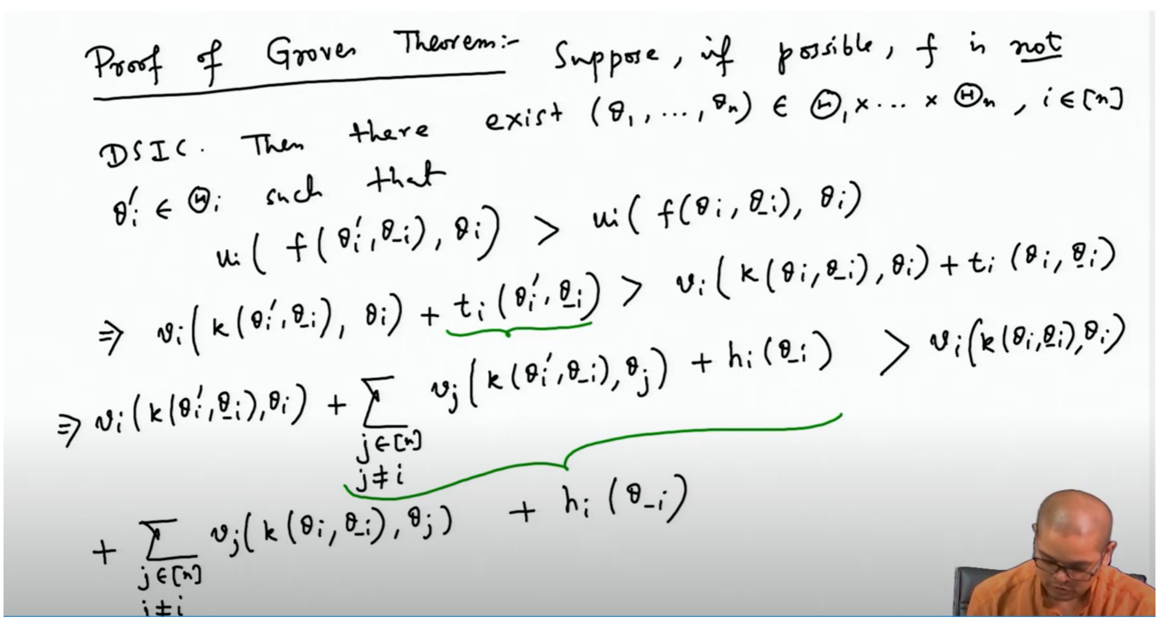
So, sum of valuations of all players this plus any arbitrary function of θ_{-i} is for all $i \in [n]$. So, groves theorem says that if your social choice function is allocatively efficient and the payment rule looks like sum of valuations plus any arbitrary function on the type profile of other players $h_i(\theta_{-i})$, then your social choice function is a dominant strategy incentive compatible ok. So, you see for payment function the because of this h_i. There is a there are infinitely many payment functions which can come with a with a allocation function k and the social choice function will be dominant strategy incentive compatible ok. So, now, before that let us first explain what does allocatively efficient social choice function mean. So, allocative efficiency.

a social choice function or let me write an allocation rule say k which takes a type profile from the from all the players and picks an allocation. An allocation rule is called allocatively efficient or simply efficient if for all type profile $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta_1 \times \ldots \times \Theta_n$. In every type profile the allocation function maximizes sum of valuations. $\sum v_i(k(\theta_i, \theta_{-i}), \theta_i)$ which is same as $(\theta_1, \theta_2, \ldots, \theta_n)$ is or this in mathematical terms this is greater than equal to $\sum v_i(k'(\theta_{-i}), \theta_i)$ and this is the sum of valuations. So, this sum of valuations is maximized by $k(\theta_i, \theta_{-i})$ this holds for all $k' \in K$. So, this allocative efficiency makes sense only in quasi linear setting, because in quasi linear setting that outcome has a structure of allocation and payment.

So, note AE make sense in the quasi-linear setting only. Now, this is allocatively efficient allocation rule. Now, a social choice function is called allocatively efficient if its allocation rule is allocatively efficient. Social choice function is called.

allocatively efficient if its allocation rule is allocatively efficient. So, now, let us go to Groves theorem. So, it says that the allocation rule is allocatively efficient and if payment has this structure then the social choice function is dominant strategy incentive compatible. So, proof of Groves theorem. So it is a proof by contradiction.

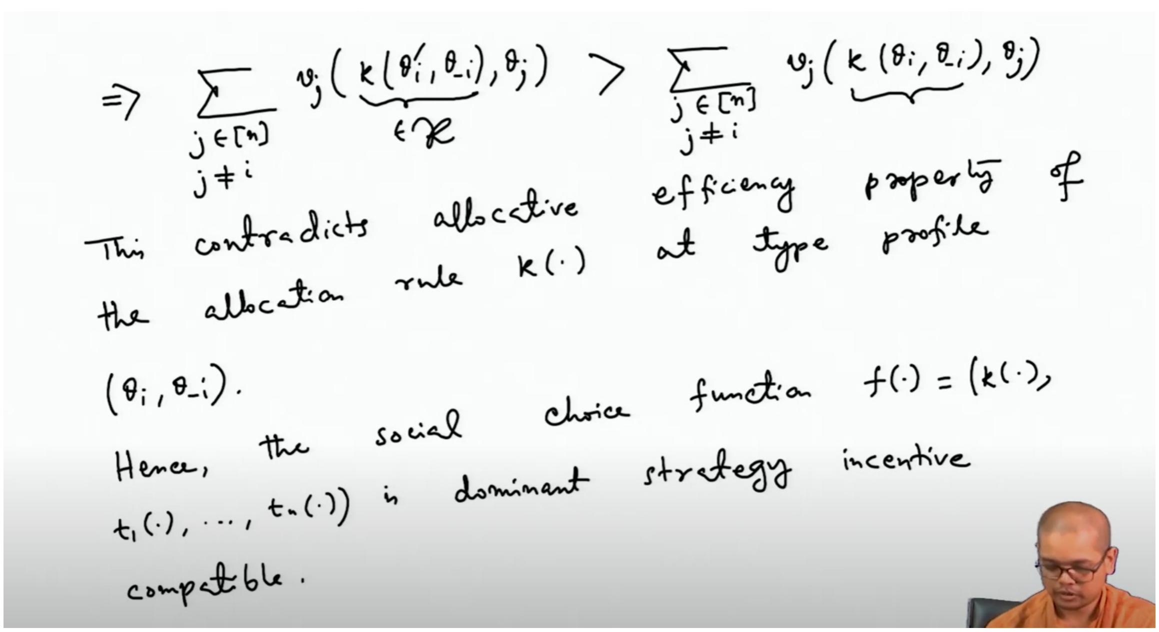
So suppose if possible, suppose f is not dominant strategy incentive compatible ok, then there exists $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta_1 \times \ldots \times \Theta_n$, a player $i \in [n]$ there exists a type profile such that there is a player and types $\theta_i^{'}$ of player i. such that from player i's perspective it is better to deviate from θ_i to $\theta_i^{'}$ it is even if it is true type is θ_i. So, there is a type profile where player i's true type is θ_i but it is better for player i to misreport her type to be $\theta_i^{'}$ that is what is meant by the this is not DSIC. such that utility of player i if it in the outcome f of $(\theta_i^{'}, \theta_{-i})$ right this is the outcome chosen by player i if every player reports θ_{-i} as usual, but player i misreports her type to be $\theta_i^{'}$ although her true type is θ_i. So, this is the utility of player i.



So, player i recall what is the utility? Utility is the function of outcome and type of player i in the quasi linear setting. It it does not directly depend on the type of other players, it depends on the type of other players via allocation rules. This is strictly greater than $u_i\left(f\left(\theta_i,\theta_{-i}\right),\theta_i\right)$. Now, we know that in quasi linear setting utilities cannot be arbitrary, it has the special structure which is valuation $v_i\left(k\left(\theta_i^{'},\theta_{-i}\right),\theta_i\right)$, this is the allocation plus payment $t_i\left(\theta_i^{'},\theta_{-i}\right)$, this is greater than $v_i\left(k\left(\theta_i,\theta_{-i}\right),\theta_i\right)+t_i\left(\theta_i,\theta_{-i}\right)$. Now, what is payment? So, we have the formula for payment here. So, we replace instead of $t_i\left(\theta_i^{'},\theta_{-i}\right)$ with this formula.

So, what is this? This is $v_i\left(k\left(\theta_i^{'},\theta_{-i}\right),\theta_i\right)$ payment is sum of valuations of other players. So, $\sum v_j\left(k\left(\theta_i^{'},\theta_{-i}\right),\theta_j\right)$ ok. This is for the player j this plus $h_i\left(\theta_{-i}\right)$. So, this is just instead of I have just replaced this payment formula that $t_i\left(\theta_i^{'},\theta_{-i}\right)$ is this this is from here ok.

Similarly, right hand side this is greater than let me write greater than $v_i\left(k\left(\theta_i,\theta_{-i}\right),\theta_i\right)+t_i\left(\theta_i,\theta_{-i}\right)$ is $\sum v_j\left(k\left(\theta_i^{'},\theta_{-i}\right),\theta_j\right)+h_i\left(\theta_{-i}\right)$. Now, this term cancels is this $h_i\left(\theta_{-i}\right)$ cancels from both side and in the sum on both side v_i is missing, but v_i is available in the in the outside of the sum. So, what we get is this $\sum v_j\left(k\left(\theta_i^{'},\theta_{-i}\right),\theta_j\right)$ this is greater than $\sum v_j\left(k\left(\theta_i,\theta_{-i}\right),\theta_j\right)$. But do you see that this violates allocative efficiency of the social choice function or in particular the allocative efficiency of the allocation rule at $\left(\theta_i,\theta_{-i}\right)$. This contradicts allocative efficiency property of the allocation rule $k(\cdot)$ at type profile at type profile $\left(\theta_i,\theta_{-i}\right)$.



Why? Because the allocation chosen is $k(\theta_i, \theta_{-i})$, but if the if you instead if you pick this particular allocation which is another allocation in the in the set of all allocations, then that will that would have increased the sum of valuations of the players. Hence, so you have found the contradiction.

the social choice function $f = (k, t_1, \ldots, t_n)$ is dominant strategy incentive compatible ok. Such social choice functions. So, this concludes the proof of Grob's theorem. Such social choice functions are also called groves mechanism ok. We also see that the payment func rules the payment rules $t_1, \ldots, t_n$ implements implements the allocation rule.

$k(\cdot)$ is DSIC or BIC if and only if the social choice function is dominant strategy incentive compatible or Bayesian Nash incentive compatible or Bayesian incentive compatible ok. So, we can see that hence Groves payment rules implements every allocatively efficient allocation rules. Hence, gross payment rules implement every allocatively efficient allocation rules in dominant strategy equilibrium ok. So, in next couple of lectures we will see we will see more specific kinds of groves allocation rules which are called Clarke mechanism and so on ok.

So, let us stop here today. Thank you.