Artificial Intelligence for Economics

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Week – 05

Lecture - 25

Lecture 25 : Network Economics (Contd.)

Welcome to the second lecture of network economics. We ended the last lecture with this particular result or proposition. So that if we talked about we were dealing with distance based utility functions where the distance where the payoff as a result of association between any two players is decreasing in the shortest distance separating them. And we saw and sees your cost of link formation. And we saw that C is less than $B_1 - B_2$, B_1 being the benefit accruing to each player, to a player who is at a distance 1 apart from any other player.

So it is the benefit accruing to a player due to an association with another player who is a direct neighbor. B_2 is the benefit accruing to a player to an association with another player who is at a minimum distance 2 apart. So, if C is less than $B_1 - B_2$, then a complete network is the is the efficient network, it maximizes the total utility of the network. If c lies in this interval, then it is a all encompassing star, that is the unique efficient network, that is what we proved in the last lecture.

And if c is greater than this $B_1 + \frac{n-2}{2} B_2$, then the empty network is the unique efficient network. great but in the last lecture towards the beginning we posited the fact that there are two essential fundamental aspects of strategic network formation one is stability other one is efficiency so when it comes to distance based utility functions or networks formed by players who abide by distance based utility functions we looked at the efficient networks what about pairwise stable networks It turns out that the pairwise stable networks are of this kind. If c is less than $B_1 - B_2$, then the complete network is the unique pairwise stable network. Well, that kind of matches with the unique efficient network. If c is less than $B_1 - B_2$, the unique efficient network is the complete network.

where it turns out that the unique pairwise stable network is also the complete network. Now if C lies between $B_1 - B_2$ and B_1, a star encompassing all nodes is pairwise stable, but it is not necessarily the only or unique pairwise stable network. And if C is greater than B_1, then a pairwise stable network either has no links, that is it is an empty network or it has at least two links, every node has at least two links, okay, great. So, let us see. So, let us try to prove this first and then we will compare this with efficiency and see where or ranges of C where we get an overlap between efficiency and pairwise stability that's the most desirable thing right we will want the efficient network to be pairwise stable but it does not happen all the time so we'll see when it happens and when it doesn't but before that let's try to prove this let's get a hang of the proof and it's fairly simple it's much easier than the last one So, if C is less than $B_1 - B_2$, then the unique pairwise stable network is the complete network, why? You can imagine this, if C is less than $B_1 - B_2$, then $B_1 - C$ is greater than B_2.

So if there are two nodes and if they are not directly connected the maximum benefit they can derive from their association is B_2. So the max and if $B_1 - C$ is greater than B_2 it means that direct connection will benefit both of them more than the best possible indirect connection which is B_2 which is separated by a distance 2. So in other words, it implies that any pair of nodes will necessarily form a link. It is rational for any pair of links to form a, any pair of nodes to form a link which in turn will result in a complete network. So that happens to be my unique pairwise stable network.

So that is number 1. What about number 2? Number 2 of the theorem states C lies in this interval. then a star encompassing all network all nodes is pairwise stable let us see. So, C belongs to $B_1 - B_2$ to B_1. So, C is greater than less than B_1 let us see.

So, if c is less than B_1 it means $B_1 - C$ is greater than 0. So, we have a star here let us say. So, $B_1 - C$ is greater than 0 the central node what is the benefit the central node is getting from any peripheral node it is $B_1 - C$. it is directly connected to any peripheral node and so the payoff to the central node due to the connection to any particular peripheral node is $B_1 - C$ and it is greater than 0. So if the central node decides to let us say snap this link then the payoff due to association with this node will be 0, because it would not be associated, but right now the payoff is positive.

So, the central node will not have an incentive to snap any link. So, the central node has no incentive to snap a link or delete a link. Now, if $B_1 - B_2$ is less than c, what does this mean? It means B_2 is or in other words let us put it this way B_2 is greater than $B_1 - C$. So, what does this mean consider any two peripheral nodes let us say this let us mark them 1 and Now what is the indirect benefit 1 and 2 are getting from each other? B_2. If they connect, if they form a link, if 1 and 2 form a link, what will be the additional benefit they will get? Well then the benefit they will get is B_1 and initially what they were getting was B2.

So B_1 is what is the new benefit which they will get if they form a link minus of course the cost of forming the new link $B_1 - C$ and initially right now what is the benefit they are getting B_2 and B_2 turns out to be greater than $B_1 - C$. So why will they form the link so it is irrational for any two peripheral links to form a link with any other peripheral node. So no two peripheral nodes have an incentive to form a new link between them. Great. And the previous condition we saw that the central node or the peripheral node.

The central node does not have an incentive to delete an edge. The peripheral node of course does not have an incentive to delete an edge. Because if it deletes an edge it will have a payoff of 0 which is horrible. great so we see that under this condition a star encompassing all nodes of the network is indeed a pairwise stable network but is it the only pairwise stable network the answer is no the answer is no.

Let us see why. Consider consider this $B_1 - C$ is lying between $B_1 - B_2$ and B_1 I am sorry $B_1 - B_2$ pardon me. So, $B_1 - C$ well let us see is so I can say that let us consider this a $B_1 - B_2$ to $B_1 - B_3$ ok. if this is the situation now if this happens this can happen right because so C belonging to this so this is a subset of right so this is fine if this happens then consider this for this network consider this network is this pairwise stable the answer is yes it is see see just just check If c is greater than $B_1 - B_2$, so any two nodes who are at a distance 2 apart will not form a new link between them. So, which nodes are at a distance 2 apart? So, let us say this and this, they will not form a link. This and this, they will not form a link.

Every other node is directly connected. So, let us make 1, 2, 3, 4 if I name the nodes. 1 and 3 are at a distance 2 apart, 2 and 4 are at a distance 2 apart. Also 1 and 2 are connected, 2 and 3 are connected, 3 and 4 are connected. If c is less than B_1 and in this case c is, we have seen that deleting an edge is irrational, right.

So, here also it is irrational. Now forming this edge between 1 and 4 is rational in this case, if 1 and 4 are not connected let us say here, if 1 and 4 are not connected then the benefit they are getting is B_3, if they form a direct connection the benefit they will get is B_1, so this is the, so $B_1 - B_3$ is the additional benefit which they are getting and what is the cost they are incurring by forming this C and the benefit is greater than C as we can see from here. so they will form this so this turns out to be a pairwise stable network but is this a star no it is not so we see that we can have other pairwise stable networks too when c lies in this interval lies in this interval so it is not a star encompassing all nodes is not a unique pairwise stable network great. What happens if c is greater than B_1? Well if c is greater than B_1 we have two possibilities. If c is greater than B_1 then what will happen? So this is number 3 now.

Well in this case $B_1 - C$ is less than 0. So consider an empty network where nobody is connected to anybody. then will any pair of nodes form a link the answer is no if they form a link what will be the benefit they get B_1 and the cost each one is incurring is $B_1 - C$ and $B_1 - C$ is negative so why will they do it right so an empty network turns out to be pairwise stable empty network is thus is pairwise stable ok also also lets if C lies between something like this then so in that case in that case yes an empty network is pairwise stable but forming a new link the benefit could be more than B_1 if it is not an empty network forming a link with any other node can yield a benefit equal to $B_1 + B_2$ let us say because it is connected to some other node for sure ok. Then it may happen that this is positive $B_1 - B_2 - C$. So in this case, a pairwise stable network can arise if all nodes necessarily have at least two neighbors.

Right? Great. So we can see that if C is greater than B_1, we will necessarily not have an all-encompassing star, which is my efficient network. if C lies in this interval $B_1 - B_2$ to B1, we will have a star which is pairwise stable, the efficient network which is a star encompassing all nodes that will turn out to be pairwise stable, but that is not a unique pairwise stable network. only if C is less than $B_1 - B_2$ then we have a complete network which is the unique pairwise stable network and that is also as we saw in the previous result the unique efficient network. So, when C is less than $B_1 - B_2$ efficiency and pairwise stability coincide. when c lies between $B_1 - B_2$ and B_1 a the pair the efficient network is necessary necessarily pairwise stable, but there are also other pairwise stable networks which are not a efficient and when c is greater than B_1 a we will never have a pairwise an efficient network great.

So we have looked at or we have talked about distance based utility models. And we have proved this. Now we'll go into another kind of networking model called the co-authorship model. Now let's understand what this is. Consider any two, we have a bunch of players again, N.

These are my players which we have. and let us say if i and j are connected if any two players i and j are connected the utility which player i derives from being connected with j is given by the utility which i derives from utility of I let me write it a little better. The utility which i derives from connecting with j that is given by This is the utility which i derives from connecting with j and the utility with j derives from connecting with i. So, let us say this is the benefit B_{ij} generated when there exists an edge between i and j. So this is the utility or benefit generated when there exists a edge between i and j.

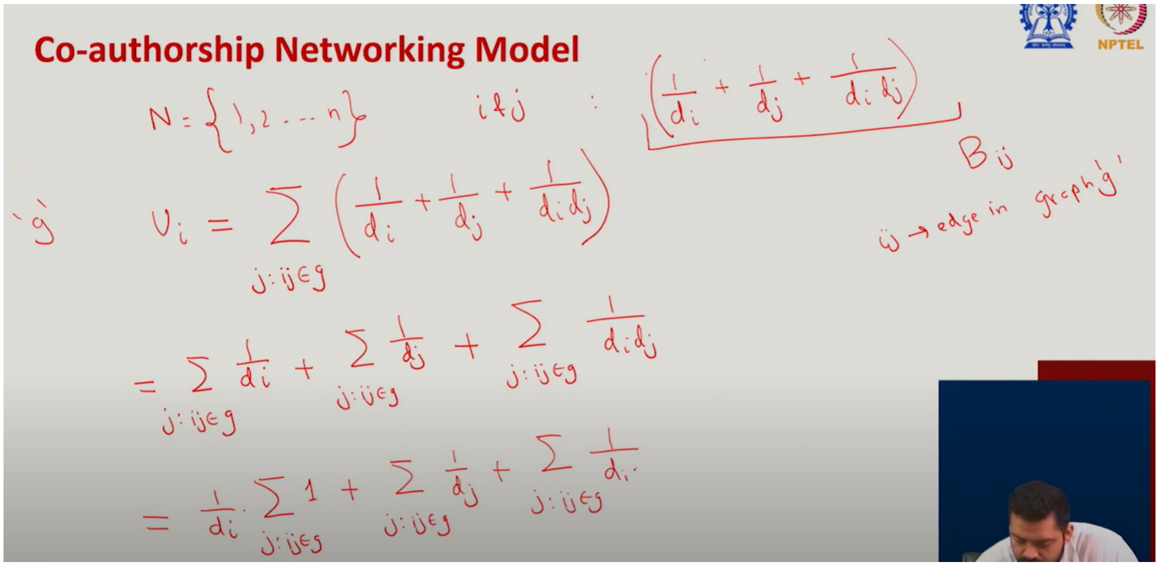
Well the benefit is inversely related to the degree of i. Why? What does it mean? What does it imply? It means if you and I are connected, the amount of benefit I will get from you and the amount of benefit you will get from me. it is inversely proportional to my degree or the number of people I am collaborating with. So let's say you and I are writing a paper or we are engaged in a project, the benefit I will get from you is more if you have lesser number of other collaborators. So if you have lesser number of other, in the best case you have no other collaborator but me, then you will devote all your energy to me.

right and then I will I will get the maximum benefit ok. So, the project or we get maximum benefit or I derive maximum benefit from you, if the you have lesser number of other collaborators or neighbors with whom you have a direct linkage Okay so this is the benefit which is generated when i and j are connected to each other. So what is the utility of i in a network G? Let's say I have a network G, what is the utility of i? Well it is simply given by summation over all these, so this is the benefit which i gets due to connection with j. but it will get these benefits from all such j's with whom i is connected. So, this is summation over all j such that an edge $\{i, j\}$ belongs to G.

So, $\{i, j\}$ is an edge in graph G ok great. Now, let us simplify this a little bit. So this is simply I forgot to mention about the last term $\frac{1}{d_i d_j}$. So, that is a factor which is additionally generated due to collaboration ok. So, if you and I are connected we have some additional benefit which is being generated apart from our own efficiencies ok.

So, this is this is this. Now, let us see let us look at the first summation. Well, this is a summation over j which are i's neighbors, but this is d_i. So, $\frac{1}{d_i}$ will come out and this is summation over j this is just 1 right. If I take the $\frac{1}{d_i}$ out I am simply left with 1. Now you can imagine what will this summation turn out to be? Well if I simply sum 1 how many times? j times such that $\{i, j\}$ belongs to G that is I will sum up 1 the number of times there is a j such that it is linked to i.

So this summation will give me d_i that is the degree of i. So I will sum up 1 the number of times i has a neighbor. So, this is simply $\frac{1}{d_i}$ into d_i plus summation over j such that $\{i, j\}$ belongs to G, $\frac{1}{d_j}$ plus again the same thing remains, great.



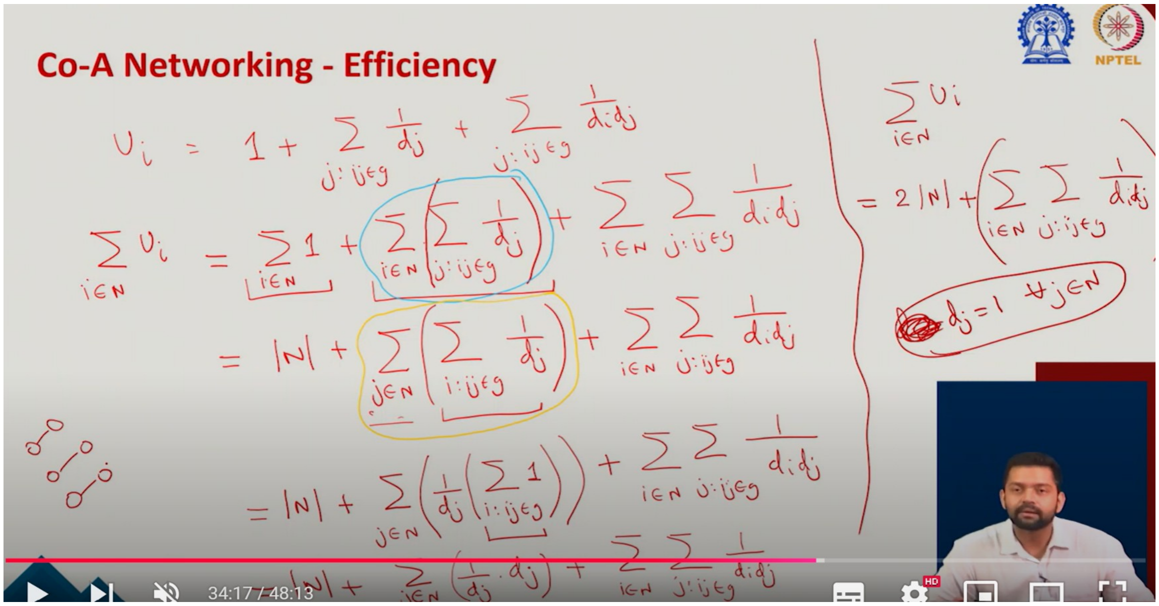
So, this is simply 1. Okay so that is the utility of I in any particular network.

Now again we are going to do this, we are going to do the same exercise. The point I want to hammer home is this constant conflict between efficiency and pairwise stability. So let us try to do the first exercise. Let us try to find out if we have this co-authorship kind of utility functions. in a network, what is the most efficient network? What network is the most efficient network? Let us try and see.

So, this is u_i. So, let me write down u_i once more. So this is my u_i, now I want to find the efficient network or in other words I want to find a network such that this is maximized. But let us first try to find this, what is the summation of utility? So this is or actually let us I think a better way of writing this is I have defined capital N as my set right. So, instead of writing this let me be a little more precise let me use this notation since capital N is my set and we have So, this is what we are left with ok. Now this of course is the cardinality of n mod n.

Now look at this summation, the second summation. What is it doing? What is this doing? What is this? I am just summing up the reciprocal of the degrees of all neighbors of i. So, I am picking any i node i and I am summing up the reciprocal of the degrees of all the neighbors of i. I am noting it down and then I am repeating this for all i's and then I am adding this entire thing up and that is what gives me the summation. Instead what if I do something else? I pick a node and I say how many times will the reciprocal of the degree of this node be added up in the summation? Well the number of times it features as a neighbor of any other node.

So what if we write it in this manner? So I reverse the order of the summation. So what



is this? What is this? This is the number of times for any particular node j. the number of times its reciprocal is added up in this in this big summation in this double summation when whenever it features as a neighbor of any other node i okay so i find out how many times $\frac{1}{d_j}$ will appear in the big sum and then i am doing i am noting it down for all j's and then adding it up so the first big sum and the second big sum so this so if i may use another color So this and let us say this. So the red and the yellow big sums are exactly equivalent right.

Now let us get back. So this is mod n. look at the summation inside now it is a summation over i's but this is $\frac{1}{d_j}$ right so $\frac{1}{d_j}$ comes out so let's see what this becomes. This summation I am keeping it unchanged. What about this? This term is simply dj again, it is i such that $\{i, j\}$ belongs to G. so $\{i, j\}$ belongs to g is an edge between i and j exists all such i's for all such i's I am having a 1 so basically how many times I am adding up a 1 I am adding up a 1 as many times as there exists an $\{i, j\}$ an edge $\{i, j\}$ with j so this becomes d_j so this is simply that's what it becomes now this is simply one again right $\frac{d_j}{d_j}$ and it is being summed over all j's so if I write it here so what is $\sum_{i \in N} u_i$ then so this is again cardinality of n so that I have two cardinality of $\frac{n+1}{d_i d_j}$.

Now take a look at this last term, this. When do you think this will be minimized? You can pause the video, sit over a cup, make a cup of coffee for yourself and try to think when will this big summation be minimized, the one at the end. It turns out the degrees are definitely greater than 1, greater than equal to 1. So, it turns out that because we are talking about connected networks here basically. So, this will turn out that this will be maximized this summation will be maximized if $d_i = d_j = 1$ or in other words if. So, the sum will be maximized if I may say actually the sum will be maximized if d_j is 1 for all j belonging to n ok something wrong with this yeah let me put this way.

right so if both the degrees are equal to 1 that's when the summation will be maximized great but what does that mean what does it mean that the efficient network is that where the degree of every node is 1 well it simply means that if you assume let's assume that there are an even number of nodes or even number of players it simply means that we'll have pairs of this kind we will have pairs of this kind, that will be an efficient network. So, if we have 10 players, we will simply have 5 pairs, this is how the efficient network will look like, in a co-authored networking model, co-authorship networking model, clear. So, it turns out that we will have a few Disconnected pairs.

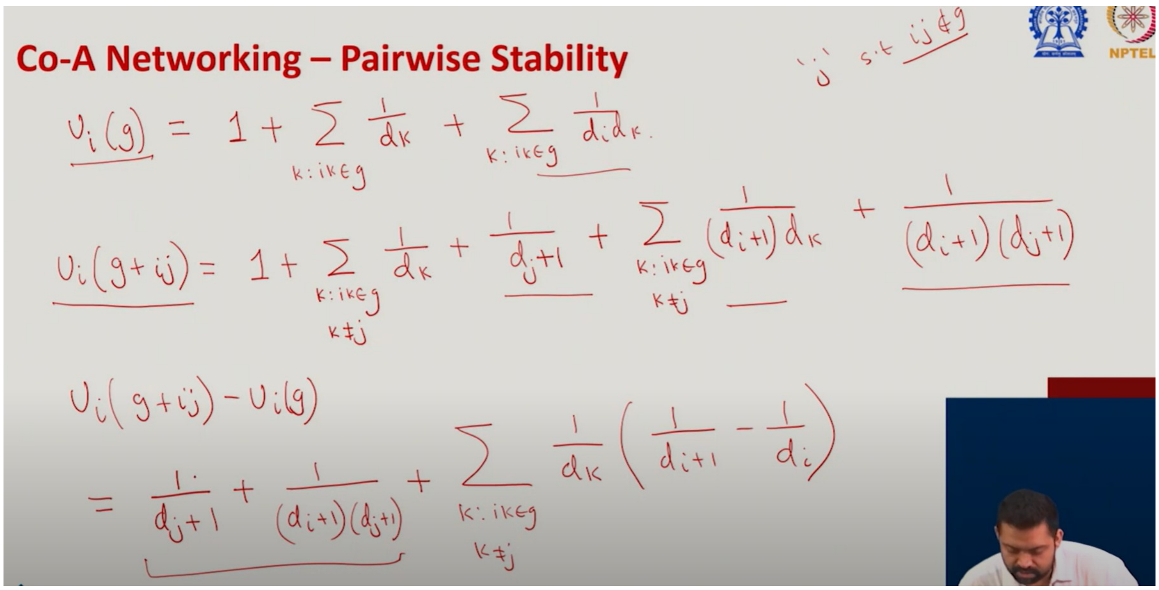
That'll be my efficient network. Great. Now let's try to understand what will be the pairwise stable network. Is the efficient network pairwise stable? Let's try to analyze that. Let's say I have a network, an agent i or a player i. What is this utility? What is it that we have seen? We have simplified this. This is one plus summation of right this is what it is now let's say there is another player j such that $\{i, j\}$ does not belong to G right now now if that's the case then what if i forms a new link with j then what will be the new payoff under the new network So let's say there is a player j, I repeat once more, where i and j are not connected in the graph g.

But if they decide to form a link, then I have a new graph, g plus ij, right? ij was not in g initially. So this is my new graph. Now what will be the payoff? Well now i has degree di plus 1. So let's see what the payoff is going to be. So it's going to be 1 plus Let's say j had degree j initially in graph G.

So now it has degree dj plus 1 because it has an additional edge with i. So this is the new payoff which we end up getting. Right? Fantastic. Now the question is, is this better than this? Does I have an incentive to form this link? So let's find out. So what is what will this turn out to be? The 1 will of course cancel out and the $\frac{1}{d_k}$ will also cancel out.

So, what we are left with is $\frac{1}{d_j+1}$ then you have, so this is there. So, this minus this, so what will that become? Of course, we also have this additional term which is $\frac{1}{d_i} + ¿ \frac{1}{d_j+1}$ plus summation over k So this is what we are left with. Great. Now let's try to simplify this a little bit.

Let's take LCM of these two. So what are we left with? We are left with d_i+2. So, we have taken an LCM. So, $\frac{d_i+2}{d_i+1}(d_j+1)$ right and then we have the summation which is k i k not belonging to g i k belonging to g I am sorry k not equal to j. So, I have taken LCM here, this is what it turns out to be, great. Now, when will this be positive? So, this is my difference in utility of i, if i forms an edge ij minus what it was having without forming that edge.

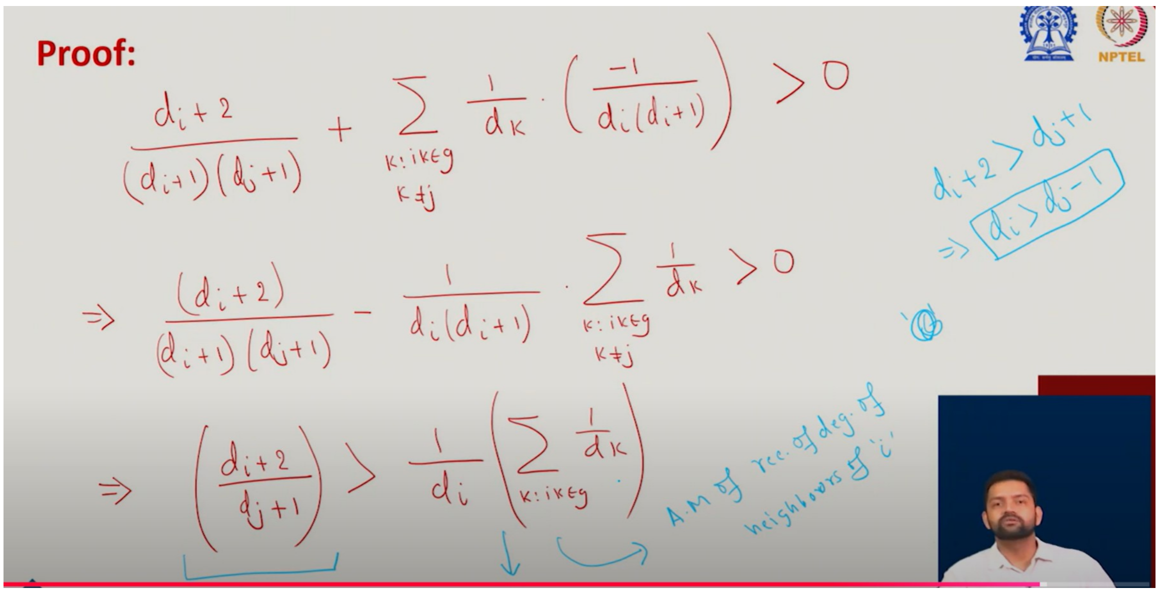


So, when will i form this edge $\{i, j\}$ with j, if this is positive? Now, when is this positive?

Great. Now let's take a look at the right-hand side. Let me use another color maybe. Let us look at the right hand side. What is this? It is the arithmetic mean of the reciprocal of the degrees of the neighbors of i. So, this is arithmetic mean of reciprocal of degrees of neighbors of i. Now reciprocal of degrees is the necessarily a fraction or at best 1.

So the arithmetic mean is necessarily less than 1. So this is less than 1. So if the right hand side is greater than 1, then the link will necessarily be formed. If the right hand side is greater than 1, then a link will necessarily be formed. So when is the right hand side greater than the left hand side? If d_i+2 is greater than d_j+1 then the right hand side is greater than 1 or in other words if d_i is greater than d_j-1 then the i always has an incentive to form i but if this happens then The right hand side is, left hand side is greater than the right hand side or in other words the inequality holds, the inequality holds. How did I arrive at the inequality? Well the inequality is basically this, remember? It is the difference between the utility which i has if it forms a link with j minus without the link.

So if it turns out to be positive, it means i has an incentive to form a link with j. So if d_i is greater than d_j-1, i has an incentive to form a link with j. So if d_i is greater than d_j-1, i will form a link with j. Similarly, proceeding if d_j is greater than d_i-1 that will mean j will form a link with i. Now, this basically this implies d_i is less than or sorry d_i is less than d_j+1.



So if d i is greater than d_j-1, i will form a link, if d_i is less than d_j+1, j will form a link with i, right, great. So $d_i=d_j$ is when both will necessarily form a link with each other, both have an incentive to form a link with each other, if $d_i=d_j$, fantastic. Which in turn means that the pairwise stable network in a co-authored networking model is where players with the same degree are always connected to each other.

Okay? Great. So we make this inference. As soon as we make this inference, let's recall what was my efficient network in the co-author network model. It was something like this, where we had independent disjoint pairs. This was my efficient network. But this efficient network can never be pairwise stable.

Why? Because look at this. this node and this node so let's say 1 2 3 4 5 6 7 8 so 1 all nodes have degree 1 here so they should be connected they have an incentive to connect to each other which means an edge like this will be formed an edge like this will be formed right so in this situation in the efficient network this is not satisfied the condition of pairwise stability is not satisfied right. So, nodes with equal degree are not connected which in other words they do have an incentive to form a new edge. So, the efficient network is a network where in this case in the co-author network model the efficient network is a network where every vertex which uh in in the so let's say any vertex in the independent network let's say independent pair has an incentive to form a link with another vertex in another pair okay hence it is pairwise unstable great so we have looked at this kind of network the quater network model network and we see that we have proven that the efficient network which we get is not pairwise stable. Great. So the motive for the last two lectures dealing with network economics and network games whatever you might call it is to give you a glimpse of this constant conflict between stability and efficiency which is what we observe in many domains of economics.

Thank you. I hope this was an interesting lecture. Thank you. See you in the next.