Artificial Intelligence for Economics

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Lecture - 23

Lecture 23 : Game Theory – Rubenstein Bargaining

Welcome to this lecture of Game Theory. In the previous lecture we looked at sequential move games and we learned the idea of sub game perfect Nash equilibrium. In this lecture we look at one interesting application of that which is Rubenstein Bargaining. Now Bargaining is something which we observe all around us. when you seek employment the employee and the employer they bargain over salary or wage you go to a market and you bargain over a an object which you are purchasing from the shop so we see bargaining happening all around between corporations between governments Now Rubenstein bargaining is a specific kind of bargaining, a specific structure which we will look into. Let us move ahead.

Let us start with the very basic form of game which we will extend as we move on. It is called the ultimatum game. Let us say there are two players who are bargaining over something, the two players are named A and B, they are bargaining over a price which is worth equal to 1 and it is common valuation, both value it equal to 1. So, that is the value of the price which the two people are fighting over.

Now, how will this bargaining happen, what distribution will finally will will the 2 players settle for. Let us say the rule is this let us say a makes an offer $(x_A, 1-x_A)$ that is a gives himself x_A I am sorry and gives b $1-x_A$ where x_A is of course, lying between 0 and 1 closed interval b either accepts it or rejects if B accepts A's offer then A gets x_A and B gets $1-x_A$ and if B rejects the offer both of them end up getting 0, ok. And the last rule is something which I have added just to make things life simple. Let us say B is indifferent between accepting and rejecting he always accepts, ok. So, if I have to present this game diagrammatically, like we were doing in the last class, last lecture.

So, A, A is making a, a player A, if you might call it so, is making an offer, which is, let us say, $(v_1, 1-v_1)$, something like this, or $(x_A, 1-x_A)$, whatever you call it. Player B, is either accepting it, in that case they end up getting $(v_1, 1-v_1)$ or rejecting it in that case they end up getting 0 and 0. If this game happens what is the sub game perfect Nash equilibrium of this game? Very simple. If A knows that B is indifferent between accepting and rejecting if B is indifferent between accepting and rejecting B accepts right. So, how much will A give B? A will try to give B as minimum as possible right as less as possible.

So, here A will naturally give B nothing. So, he will give B $1-v_1$ will be equal to 0 ok v_1^*. So, v_1^* will be 1 and $1-v_1^*$ will be 0. So, if this negotiation happens then A will get everything and B will get nothing. This is the one shot proposal game A is proposing B can either accept or reject and when B is indifferent between accepting and rejecting he accepts.

In this situation A will definitely make this terrible offer he will keep everything for himself and give B nothing. That is the sub game perfect Nash equilibrium of this game, great. So, in this situation we see that whoever proposes, whoever offers has all the power, he gets a massively good deal, in this case it is A. What if we add another layer to the game? ok let us see how. Let us say player 1 which is a let us say player 1 is making an offer $\left(v_1, 1-v_1\right)$ what was player 2 or B what was he doing he was either accepting if he accepts then they end up getting this payoff $\left(v_1, 1-v_1\right)$ or rejecting. Now, if player 2 rejects, then let us say the following happens.

Imagine the situation, how is the this how is this negotiation taking place? Let us say it is taking place in a in a hotel room. So, player 1 and player 2 have gathered in a hotel room today and they are trying to negotiate. Player 1 has given this offer $\left(v_1, 1-v_1\right)$, if player 2 accepts the deal is struck they take they accept this distribution and move on in life or player 2 rejects. Now what happens? If player 2 rejects player 1's proposal then they say that ok fine you have rejected my proposal we will meet again let us say one month after and then so they meet one month after in the next period and then player 2 makes a proposal. So, we have added another period.

Now player 1 can either accept it or reject it. If player 1 rejects it then both of them get 0 and 0. If player 1 accepts it then what will be the payoff? Well the payoff should be $\left(v_2, 1-v_2\right)$. but there is a small change it will be $\left(\delta\left(v_2\right), \delta\left(1-v_2\right)\right)$ what is this δ? This delta is what I call the discount factor what is where did this come from very simple See, if you and I are negotiating and trying to strike a deal, it's better to strike the deal now rather than postponing it in the future, right. 10 rupees now is much better than 10 rupees one year from now, right.

So postponing your benefits to the future or postponing your benefits to subsequent periods, that has a cost, it reduces your benefit. by how much by an amount δ which is called the discount factor and delta is a fraction lying between 0 and 1. So, they end up getting this $(\delta(v_2), \delta(1-v_2))$, if player 1 accepts it, if he rejects it then well it is 0 0 fine. Now, you have this game tree can you guess you can pause the video and try to find out what will be the sub game perfect Nash equilibrium of this game let's see let's start at the end and then we will move up we will move back in time that's backward induction this is player 1 what should player 1 do should he accept or reject well if player 1 rejects he gets 0 if he accepts he gets this much and $\delta(1-v_2)$ will be either 0 or something positive if v_2 is 1 then it will be 0. So, it is definite that player 1 will definitely accept.

Now, if player 1 is accepting what should player 2 choose which maximizes his payoff? Well, he should simply choose $v_2^*=1$ and $1-v_2^*=0$ and this should be 1 right. He should give player 1 nothing remember the first payoff is that of player 1 and the second payoff in the payoff vector is that of player 2. So, he should give player 1 nothing and keep everything for himself ok. Now, if P_2 offers this player 1 accepts what is their payoff here? Player 1 will get 0 and how much will player 2 get? $1\delta(1-v_2)$ which is delta right ok. Now, think about it player 1 has made an offer player 2 is rejecting let us look at player Player 2 knows that if he rejects and this game, let me use another color ink maybe and if this game happens, this blue game happens then he will get a payoff of delta, right.

Then what should, so player 2 if he rejects he is going to get delta for sure. Now how much should player 1 offer here? Well, player 1 will offer exactly as much which will make player 2 indifferent between accepting and rejecting, right. So, he will offer v_1^* in a way such that $1-v_1^*$ will be equal to delta or in other words v_1^* is going to be $1-\delta$. So, this constitutes my sub game perfect Nash equilibrium v_1^* is $1-\delta$ and $1-v_1^*$ is δ and we have v_2^* is 0 and $1-v_2^*$ is 1. So, this is what the sub game perfect Nash equilibrium is and as a result of the bargaining what is going to happen what is happening.

Basically this is my result of the bargaining this is how the sharing will happen. Given that player 1 knows that if the game enters this blue sub game then the payoff is going to be this 0 δ that makes player 1 make an offer of this kind of this kind. okay. And this offer is necessarily accepted because player 2 also knows what is going to happen in the blue sub game and he is getting exactly as much as he would get if he rejects and enters the blue sub game, okay. So, this is how the distribution happens.

Player 1 gets $1-\delta$, player 2 gets δ. Now, we see there is a significant improvement. How much was player 1 getting initially? When he was the only offer he was the only one offering he got 1 and player 2 got 0. Here he is getting δ and player 1 is getting $1-\delta$.

So, when player 2 got some additional power of counter offering his payoff went up from 0 to delta and player 1's payoff went down by delta.

Great, that seems interesting, it means that this counter offering, this power to counter offer, this option to counter offer leads to better payoff. Great, let's add another period and let's see what's happening. Let us see what is happening, let us add another period, so let us say we have P_1 player 1 making this offer, player 2 either accepts it or rejects it, if he rejects it he offers he proposes this distribution, player 1 accepts it, then they are going to get $(\delta(v_2), \delta(1-v_2))$ or player 1 rejects it. And now player 1 makes another counter offer which is $(v_3, 1-v_3)$. Now, player accepts it, if he accepts it this is the third period now.

So, the payoff is going to be $\delta^2(v_3)$ $\delta^2(1-v_3)$ and rejects it, it is going to be 0 0. So, we have just added another period to the game. Now let us try to see what is going to happen. Let me use another ink just to make things a little easy. Now consider P_2 here, this is P_2.

What should P_2 do? Well, player 2 if he rejects he is going to get 0 and if he accepts he is going to get this much. square 1 minus V 3 is definitely non negative. So, and if he is indifferent between accepting and rejecting what will any player do? Accept. So, P_2 necessarily accepts here. If P_2 accepts what is player 1's this is player 1 here what is player 1's optimal offer? Well, player 1 knows that P_2 here will definitely accept.

So, player 1 will keep everything for himself and give p to nothing, which means v_3^* will be 1, $1-v_3^*$ should be 0, ok. Very good. Now, now look at player 2 here. Player 2 knows that if player 1 rejects his offer whatever he is offering then this red sub game will take place. This red sub game will take place the one which has been marked by this dotted line and in this red sub game how much will player 1 get? He will get v_3 which is basically $\delta^2(v_3)$ because it is in the third period right.

So how much should P_2 offer, player 2? Well he should offer exactly as much which will make player 1 indifferent between accepting and rejecting. Why offer more? That is irrational. So he will offer in a way such that this $\delta(v_2)$, so he will make this offer $(v_2, 1-v_2)$ such that $\delta(v_2)=\delta^2(v_3)$. So, $\delta(v_2)=\delta^2(v_3)$ which means or $\delta(v_2^*)$ if I may call it so. So, v_2^* isδ , but what is v_3? v_3 was 1.

So, this simply is so v_2^* is going to be simply δ is 1 great. So, this is the optimal offer v_2^* being δ. Now, if now let us look at player 1 here, let me use another ink now, let us use a green ink. Now, player 1 knows that if player 2 rejects then the game will enter this larger green sub game.

and in this green sub game player 2 will make this offer $(v_2, 1-v_2)$ where v_2^* is δ. If v_2^* is δ then how much is player 2 getting he is getting $\delta(1-v_2)$ which is how much $\delta(1-\delta)$. So, if P_2 rejects he is going to get 2 player 2's payoff is going to be $\delta-\delta^2$ ok. So, if that is the case then how much should player 1 offer player 2? Well he should offer offered player 2 exactly as much which will make player 2 indifferent between accepting and rejecting why offered more. So, $1-v_1^*$ should be equal to $\delta-\delta^2$ or in other words v_1^* should be $1-\delta+\delta^2$.

Now, compare it with the two period game. In the two period game the payoff was the first player got $1-\delta$, the second player got δ. here the first player again had an added counter offer opportunity at the end here, right. So, now his payoff becomes $1-\delta+\delta^2$ and what about $1-v_1^*$ which is player 2's payoff it becomes $\delta-\delta^2$.

So, initially it was delta. So, player 1's payoff has gone up by δ^2, player 2's payoff has gone down by delta square. due to this counter offering extra counter offering liberty or opportunity which player 1 has got, which means counter offering is a great power, okay. An additional period of counter offering if allowed that gives that particular player who can counter offer a much better deal at the beginning in the sub game perfect Nash equilibrium. Which means what? Which means that everybody will both players will want to counter offer indefinitely. They would and player 1 knows that player 2 will want to counter offer indefinitely.

It is rational for both players to keep on counter offering day after day, month after month, year after year. right, great, then what will be the sub game perfect Nash equilibrium in such a situation, let us understand. If, so what did we see, if it is a one stage one game, if it is a or let us say I call it a period one game, one period game which was basically my Just a sec, I think I should just add a few more slides here. So in the one period game, what was the, in the one period game, what were the outputs, what were the payoffs? v_1^* was 1, $1-v_1^*$ 0, player 1 got everything. If it is a 2 period game, where player 1 offers and then player 2 counter offers, then v_1^* became $1-\delta$, $1-v_1^*$ became δ.

If it is a 3 period game, then what happens? So, player 1 offering, player 2 counter offering, then player 1 again counter offering. So now see player 1's payoff goes up and player 2's payoff goes down by amount delta square, right. So it goes on like this. Now if it goes on infinitely what is going to happen? What will be v_1^* if for infinite number of periods? let me denote it like this v_1^* infinity. Well it will be simply this alternating $1-\delta+\delta^2-\delta^3+\delta^4-\ldots$.

Well this you can simply separate out the 2 geometric progressions which are easily visible. So, this is $1-\delta+\delta^2-\delta^3+\delta^4-\ldots$ and remember δ is a fraction. So, all the common ratios of all the GPs or both the GPs are fractions less than 1 positive fractions. So, this is simply $\frac{1}{1-\delta^2}$ this is $\frac{\delta}{1-\delta^2}$. So, this is $\frac{1-\delta}{1-\delta^2}$ that is $\frac{1}{1+\delta}$ ok.

So, v_1^* is $\frac{1}{1+\delta}$ and $1-v_1^*$ if infinite counteroffering is indeed happening it is $\frac{\delta}{1+\delta}$, great. So, it means that if both the players know that there is no restriction when it comes to counteroffering then counteroffering will indeed go on indefinitely and the players will end up getting Players at the on the very first day will choose to settle for this $\frac{\delta}{1+\delta}$ this offer. Player 1 will just make this offer and it will be optimal for player 2 to accept ok. So, that is the that is the distribution which they will settle for great. Now, there is another way of looking at this game.

I mean the way we solved it is fine, but there is a different way it's usually solved, so just for the sake of completeness I will cover that. So this was our game, so player one is making an offer, player two either accepts it and they get $(v_1, 1-v_1)$ or rejects it and then player 1 sorry player 2 again makes an offer now v_1-v_2. player 1 accepts it and they get $(\delta(v_2), \delta(1-v_2))$ or player 1 rejects it. Now, when player 1 rejects player 1 again makes an offer. So, this is where and it goes on let us say this is how the game is and let us say.

In this infinitely repeated game, infinitely repeated offering counter offering game which is called Rubenstein bargaining, let us say the sub game perfect Nash equilibrium leads to v_1^* equal to v^*, v_2^* or $1-v_1^*$ is $1-v^*$. Let us say this is the outcome of Rubenstein bargaining game, the sub game perfect Nash equilibrium outcome at the beginning, okay. if that is the case if that is the case then if player 1 rejects this is a Rubenstein bargaining game starting here starting at this point starting at this point it is a Rubenstein bargaining game. So, what should be the what should be the sub game perfect Nash equilibrium outcome of this it should be $(v^*, 1-v^*)$ starting here. So, now player 1 knows that if he rejects in the next period he will get $(v^*, 1-v^*)$ actually that will be the third period.

So, he will basically get $\delta^2 v^* \times \delta^2(1-v^*)$ right. So how much should player 2, so let me write in this part of the board now. So how much should player 2, so how much should player 2 propose? Well, player 1, player 2 will propose or give player 1 exactly as much such that he is indifferent between accepting and rejecting, anything more will be irrational. right if he accepts he gets δv_2 if he rejects he gets $\delta^2(1-v^*)$ right. So, δv_2

should be equal to $\delta^2 v^*$ or in other words v_2^* should be equal to δv^* right ok.

Now, think about this sub game, the black sub game now. If player 2 rejects, he is going to get how much? He is going to get $\delta \times (1-v_2)$, right. So, player 1 when player 1 is making an offer, how much should player 1 offer player 2? what should be player 1's offer here $(v_1, 1-v_1)$ it should be exactly as much such to such as he provides player 2 an amount so so that he is indifferent between accepting and rejecting right anything more will be irrational for player 1 to offer and anything less will lead to rejection or rejection and thereby payoff of this so what is $1-v_1$ then so $1-v_1^*$ which is what player 2 gets should be equal to $\delta \times (1-v_2^*)$ great. So, what is what is v_1^* here, but what is v_2^* by the way it is δv^*.

ok great. So, what is v_1^*? Let us let us solve it at this end of the board. So, what is v_1^*? v_1^* turns out to a $(1-\delta+\delta^2)v^*$ fine now observe something well this was a Rubenstein bargaining game yes but is in the entire game a Rubenstein bargaining game starting here starting at this point the answer is yes absolutely so what should be the sub game perfect Nash equilibrium of the total Rubenstein bargaining game well it should be v_1^* equal to v^* which means this should itself be equal to v^*, right, great. So, if that is the case then what is that? If I substitute v^* or v_1^* with v^* then what should we get? Let us see. So, v^* is simply equal to then $(1-\delta+\delta^2)v^*$. then v^* is simply $\frac{1-\delta}{1-\delta^2}$ which is $\frac{1}{1+\delta}$.

Oh and that is exactly what we had got in the previous slide where we had solved it in a different way, right v_1^* was $\frac{1}{1+\delta}$ and that is exactly what we have got here as well as a solution of the Rubenstein bargaining game. Fantastic, so we have looked at two ways of solving the Rubenstein bargaining game. Now a little extension of this game and a very interesting insight of conclusion. Let's imagine if the discount factors which we have that is what is the discount factor? How did we interpret it? it is the amount by which my payoff is or the factor by which my payoff is getting diminished if I postpone or delay the deal, okay, so if I keep rejecting, if a player rejects and there is another round of negotiation happening in the future we are postponing the deal into the future and that has a cost that is that cost is happening and delta captures that cost that's what we call the discount factor but there are two players both players it's not necessary that both players will hate waiting equally waiting is bad we would want the deal to be struck as early as possible, but waiting reduces my payoff by a certain factor delta, but that factor may be different for both the players right.

Let us see, let us say the 2 players, player 1 has just a second let me change the color

Player 1 has δ_1 and player 2 has δ_2, these are their discount factors. And of course, both the deltas are fractions, positive fractions, okay, great. So, what is the game? What is going on? Let us again start with solve the Rubenstein bargaining the way we just did, but with different deltas now. So player 1 is making an offer $(v_1, 1-v_1)$. I am just repeating the process which we just did in the last slide.

Player 2 either accepts it and gets a payoff of what is going to be the payoff $(v_1, 1-v_1)$ or rejects it. If he rejects he makes an offer. Now, player 1 accepts it or rejects it. If he accepts it, then this is the second period. What will be the payoffs? Player 1's payoff is going to be $\delta_1 v_2$ and player 2's payoff is going to be $\delta_2(1-v_2)$.

And if player 1 rejects then what is going to happen? Then player 1 again makes an offer, player 1 again makes an offer and we are again in that Rubenstein bargaining game. And let us say the solution of the Rubenstein bargaining game, the sub game perfect solution of the Rubenstein bargaining game is v^*, okay. and the payoff is going to come in the next period after p_1 proposes. So, the payoffs are going to be $\delta_1^2 v^*(1-v^*)$.

Okay? Great. Now, now let's solve it exactly the way we did in the slide before. What will, what will P_2 propose here? Well, P_2 is going to propose in a way such that P_1 is indifferent between accepting and rejecting, that is, $\delta_1 v_2^*$ that is the optimal v_2 should be equal to $\delta_1^2 v^*$ which is the solution of the Rubenstein bargaining game. Great which means v_2 is v_2^* is $\delta_1 v^*$ very good. So, this is what P_2 offers. Now, if that is the case then in this red sub game what is the payoff the 2 players are getting? So, v_2^* is going to be $\delta_1 v^*$.

So, what is player 1's payoff and player 2's payoff? Player 1 gets or player 2 gets how much? Player 2 gets $\delta_2(1-v_2^*)$. this much ok and how much is that? So, this is $\delta_2(1-\delta_1)v^*$ that is what player 2 gets if the red sub game is entered ok. So, how much should player 1 offer player 2? How much should player 1 offer player 2? well if he he should offer in a way such that player 2 is indifferent between accepting and rejecting here if he enters the red he is going to get this much and if he accepts if he gets more then that is that is that is stupid from player 1 so player 1 should offer this in a way such that $1-v_1^*$ is equal to $\delta_2(1-\delta_1)v^*$ ok great. So, that is the case now we know that this is again the start of the Rubenstein bargaining game a larger Rubenstein bargaining game which is this and since we have initially assumed that in a Rubenstein bargaining game with where infinite offering and counter offering is happening $v^*(1-v^*)$ is the distribution which is taking place.

So, v_1^* should also be v^*. So, I will replace v_1^* with v^* ok. Great and then what is v^* then? v^* is simply $\frac{1-\delta}{1-\delta_1\delta_2}$, okay. In the previous case both δ_1 and δ_2 was δ. So, it led to $\frac{1-\delta}{1-\delta^2}$ which was $\frac{1}{1+\delta}$, but in this case this is the right expression. Now, notice something interesting what is player 1's payoff in this Rubenstein bargaining $\frac{1-\delta_2}{1-\delta_1\delta_2}$

So, v^* is what can we observe from this. So, this is player 1's payoff in sub game perfect Nash equilibrium in Rubenstein bargaining. Now, what can we observe from this? Clearly what is $\delta v^* \delta_1$? If δ_1 goes up for any particular δ_2, then what happens? Then what happens? The denominator goes down, so the entire thing goes up, so this is positive. What about, what is this? How does the, how does player 1's equilibrium payoff vary with the discount factor of the other player? If his own discount factor goes up, his equilibrium payoff goes up. What about the other persons discount factor? If it goes up what will happen? Let us just simply differentiate this. Now, let us simplify this little bit, let us see what we get and we know that δ_1 is a fraction.

So both the numerator and the denominator are positive, so this is negative. So we see that the equilibrium payoff of the first player goes up if his own discount factor goes up and it goes down if the discount factor of the other player goes up. So what does it mean? It means that more patient you are, a better deal you are going to get. in the Rubenstein bargaining and more patient your opponent or your competitor is lesser will be your payoff, okay. So Rubenstein is basically unequivocally advocating patience, so this is an advertisement for patience if you may call it so.

That is the story which pops out, okay. Great, so we have learnt a way or a mechanism of bargaining called Rubenstein bargaining And before that, we learned about sequential games. Great. See you in the next lecture. I hope you enjoyed the game theory segment. Thank you.