Artificial Intelligence for Economics

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Lecture - 22

Lecture 22 : Game Theory - Sequential Games

welcome to the next lecture of game theory so in the first four lectures in game theory we have looked at simultaneous move games we have looked at pure strategy Nash equilibrium looked at mixed strategy Nash equilibrium then we started studying Bayesian games which is games under incomplete information when the type of the players are not known to the other players but they have a belief about it Today we will start with something new, we will start with sequential games. Till now the players were choosing their actions or strategies in case of a mixed strategy simultaneously but not all situations are like that. At times in the real world, we encounter scenarios where players move sequentially. So if you and I are playing a game, you move, I observe your move and then I move accordingly. So that's the kind of scenario which we will try to model in today's lecture and the lecture which is coming up. Sure, let's begin.

Before we jump into formalizing, what we'll do is, I'll present you with a little, a popular mathematical puzzle. I would urge all of you to pause the video and try and solve this mathematical puzzle before looking at the solution. Okay. Now let's go ahead and try to solve this problem.

Let's first read the problem carefully. So there are five pirates. It's usually called the pirate gold coin problem. And let's say the pirates have managed to gather a loot, 100 gold coins. The pirates are named A, B, C, D, E.

A is the senior most pirate, then B, then C, then D and E is the junior most guy. Now how will these 100 gold coins be distributed amongst the pirates? Well, these are the rules of distribution. First, the senior most pirate, which is A, he'll propose a distribution of the coins. Okay? How will a distribution look like, by the way? It looks like something like this. Let's say, I'm sorry.

it looks like something like this. So, let us say he says it is a 5 dimensional vector. So, let us say he says 20, 10, 0, 30 and 40. Okay, so this could be a, this is a possible proposal where he says that A which is himself he is going to get 20, B will get 10, C will get 0, D will get 30, E will get 40. This is a possible proposal, I am not saying that this will be the proposal.

Okay, so it is a proposal, it is basically a 5 dimensional vector such that the elements when added up adds up to 100. Okay, great. So you can write down the set of proposal in this manner, it is any vector or let us say xA, xB, xC, xD, xE such that $\sum x_i = 100$, okay. So set of all possible vectors and of course all the x_i's integers, they belong to the set of integers, okay, great. Now if this is the situation, the senior most pirate proposes or puts forward a proposal.

All the pirates, this is the second rule, all the pirates vote on whether to accept this distribution or not, this proposed distribution or not, okay. The distribution is approved if at least half of the pirates agree including the proposer. So, everybody votes including the proposer. If the total number of votes in favour is equal to or greater than 50 percent of the total votes which have been cast then we say that the distribution has been approved or accepted. If the distribution is accepted then what happens? Well, then the coins are distributed according to that distribution and the game ends.

If not then the the proposer is is is killed ok. The proposer is killed he is he is thrown out of the ship and he is killed and the next most senior pirate makes a new proposal and the process continues ok. great now what is the objective of any pirate every pirate wants to survive right you don't want to be killed second given that given that he survives he wants to maximize the total number of gold coins he will receive ok great let's move on so how will you solve the puzzle who is going to propose first A is going to propose first what do you think A should propose in this in under these under these rules let's see how to solve this problem so we solve this problem thinking backwards, let us see, let us see what I mean by that. Let us say A B C D E these were my pilots, let us say A B C have already been killed only D and E are left, let us say. Then D is the senior most, now D is proposing and then D and E will both vote for the in favor or against that particular distribution proposed by D.

If that is the situation can you guess what what is D's optimal proposal? Well the answer is simple D will simply propose D will simply propose of course, he will not give a b c anything because they are they are dead he will give himself 100 and he will give E 0. why because there are only two voters how many votes should the proposal need in order to get approved there are two votes what is fifty percent of two one so just getting one vote is enough for the proposal to go through so no matter what proposal d gives it will necessarily be approved because there are two votes and d has one vote and D will definitely vote for himself. So D will propose this and he will not give E anything even if he keeps 100 of all the gold coins with himself even then the proposal will go through great. Now think about the situation where A and B are dead and C D E are alive okay. Now who is making the proposal? C.

Now E knows that if C dies, then the first scenario will happen. And if the first scenario happens, then E will get nothing. Now C knows this. C knows that if he is dead, E will get nothing in scenario 1. so what should be C's proposal well C will say that in order to get E's vote I just need to give him one gold coin because if I die if he votes against me because D will necessarily vote against C because if C dies D is going to get everything so C is going to just do this of course there is no point giving A and B there they are dead So, C is going to keep 99 for himself, give D nothing and give C, give E 1.

So, E now gets 1, E gets 1. Now, will C of course, so there are 3 votes now, right, C, D, E. Now, will C vote for it? Of course, C is the proposer. So, C will definitely vote for himself, there is no doubt about that. will E vote for this the answer is yes because D is voting against C if E also votes against C then the proposal will be rejected and C will be dead and we will land up in scenario 1 which is very bad for E pirate E so pirate E will vote for this so this proposal will get 2 votes C and E it's 2 greater than 50% of 3 the answer is yes so this will go through so this is the situation then C will propose this this distribution now let's say let's imagine only A is dead we have B C D E then what will happen if In the CDE scenario, who is the one who is completely losing out? It is D.

D gets 0, right. A and B are anyway dead. So, that is okay. So, D gets 0 in the CDE scenario. Now, B is proposing. What will be B's proposal? B knows that if he dies, we land up in scenario 2 and D will get nothing.

so B knows that just by giving one gold coin to D he can get D's vote right and there are four voters here four voters who are alive B will definitely vote for himself so if he can get D's vote which he can get by just giving him one gold coin then the game is done then his proposal will go through so B only has to give of course nothing to A, A is not there So, he gives himself 99 gives C nothing gives D 1 and gives E nothing as well. In this situation what happens? D votes for B, B himself votes for himself B votes for himself. So, B ends up getting 2 votes out of 4 that is 50 percent and the proposal goes through. So, this is B's optimal proposal. If the situation B, C, D, E arises, this is B's optimal proposal.

Very good. Now imagine if, imagine the scenario which is there right now. Everybody is alive. Now it's A's turn to vote, A's turn to propose. A knows that if he dies, how many votes will A need? If there are 5 votes and A has to get at least 50%, so A basically needs 3 votes. Now, A can understand that if he dies, who are the people who are losing out completely? It is C and E.

C and E are getting nothing if A dies, right. Then this proposal in scenario 3 will go through. So, A will simply give this proposal, 98 for himself. 0 to B he will give 1 to C 0 to D and 1 to E. So, giving 1 to C and E will be will get A their votes.

So, A will get 3 votes and that is good enough. So, this is the optimal proposal which A will come up with. Okay, great. So this is just to give you an idea of how to think backwards. This is what we will formally define as backward induction later on.

But now that we have got an idea about how sequential games are kind of, let us formally get into it. Let us look at a game, let us look at a sequential move game, a very simple one. a kind of game which you will encounter in any game theory textbook. Let's say there are two players, two firms or two companies, an entrant and an incumbent. An incumbent is a company who is already existing in the market, okay.

So an entrant can have two possible actions which he can take. He can either enter the business in or keep out of the business. okay if he keeps out of the business then the entrance payoff is 0 and the incumbents payoff is 10 the incumbent is already there in the market right so the entrant chooses to not enter then the entrant gets nothing the incumbent gets all the market which is 10 very good if the entrant chooses in chooses to enter then then the incumbent has two possible actions accommodate or fight. If the incumbent fights then there is they will cut prices and there is business war and it will lead to losses for both of them. So, let us say they get a payoff of minus 5 and minus 5 and if the incumbent chooses to accommodate the entrant then the two of them coexist they cohabit the in the space and they share the market they get 5 and So, this is an example of a sequential move game.

Here the entrant moves first, the incumbent can see what the entrant has done and he moves after that, after having observed the entrance action. Now, some formal definitions. So sequential move game, also called an extensive form game, are characterized by these five features. First, of course, we have a set of players.

No game happens without players. Of course, we have a set of players. We also have a history. or many histories actually. What is history? A history is a sequence of actions chosen by different players. For example, in this game you have entrant has chosen in, this is a history, this is a history.

The entrant has chosen in and the incumbent has chosen fight, this is a history. the entrant has chosen in, the incumbent has chosen accommodate, so I haven't mentioned this here, so the incumbent has chosen, sorry, the entrant has chosen in and the incumbent has chosen accommodate, that's also history and then you also have the start history where the entrant moves, where nobody has taken any actions, okay, so start, that's also another history. So, these are my histories ok, great. What is a terminal history? A terminal history is a sequence of actions such that it is not a sub history of any other sequence or in other words that is where a chain ends. Example in fight or in accommodate and out right.

So, let us say you have in accommodate. Is there any other action which can follow after accommodate? The answer is no. So this is a terminal history. In fight, is there an action which will follow after fight? No. So in fight constitutes a terminal history.

Great. What is a player function? A player function is a function which assigns a player to a history. For example, when the history start who is the player assigned to that who moves at the start the answer is the entrant. So, P start is entrant start in now if after in who moves it is the incumbent ok after start and out who moves after out nobody moves. So, it is 5 which is a null set. Anyway and for each player there is a payoff associated with each terminal history, you can see for in accommodate there is a payoff, for in fight there is a payoff, for out there is a payoff, great.

So this is, these are the definitions associated with a sequential move game. Now what is strategy of a player in a sequential move game? Unlike in a simultaneous move game, a strategy of a player in a sequential move game is a function that assigns to each history h a particular action chosen from the set of actions which player i can choose. So, if after history h it is player i's turn to move any action chosen by player i which is permissible after history h constitutes or sequence of such actions constitutes a strategy. So, a particular player will move at different junctures in the game and whenever it is that players turn to move he is going to choose some action the sequence of such actions for different histories constitutes a strategy.

I will define it in the next slide. Consider the game which you were dealing with for example, this game. So, consider entrant, entrant moves only once at at start right at start. So, what is the strategy of the entrant? It is either in or out ok. what about incumbent when does he move he moves when the history is start in see start in that is when the incumbent moves and what are the possible actions which the incumbent has after the entrant has chosen in it is accommodate fight okay let us look at one more game and define the possible strategies of the players Let us say player 1 moves and chooses L or R, then player 2 moves here also its player 2 moving. Player 2 moves, player 2 chooses either U or D.

If player 2 has chosen player 1 has chosen L and player 2 has chosen U, then player 1 moves again and chooses either L or So, this is the game. Now, what are the possible strategies how to define the strategies of the different players in this way. When is player

1 moving? Player 1 is moving for what history will player 1 move? Player 1 will move when the history is start that is here or if the history is l u. right if the history is start l u.

These are the two times when player 1 moves ok. So, see the histories for which player 1 can move are either start or start l u. So, what are the possible strategies? It is he chooses l for start and l for start l u. l for start and r for start l u r for start l for start l u r for start r for start l u ok. Now, look at player 2 when can player 2 move when the history is either start l or start h sorry start r these are the 2 histories under which Player 2 will move, start L and start R and the possible actions are U and D. So under history start L or start R he can move and what are the possible strategies? U U, U D, D U, D D.

Great. Now that we have formally defined strategies, now what are the optimal strategies? Well optimal strategies are chosen by a process called the backward induction. Backward induction is a process of reasoning backwards in time like we did in the case of the pirate gold coin problem. It is a process of reasoning backwards in time from the end of a problem or situation and determine the sequence of optimal actions. So we will start at the end and move up. It proceeds by examining the last point at which a decision is to be made and then identifying what action would be optimal at that moment.

So, we start at the end and then move back in time. We will see how to apply that. Let us look at this game and let us try to solve. Let us try to solve this. Let us say the entrance chooses in and the incumbent is choosing either accommodate or fight. So, what are the optimal strategies of the different players? Let us see.

If the, so we will start at the very end. So, we will start with the incumbent. So if the entrant has chosen in, what will the incumbent do? If the incumbent chooses to accommodate, then his payoff is going to be, remember the blue payoff is that of the incumbent. So if he chooses accommodate, he gets a payoff of 5, if he chooses to fight, his payoff is minus 5. So what should the incumbent do? Well, he should definitely accommodate.

So this is what he does, accommodate, right, okay. Now what should the entrant do? The entrant now knows if he chooses in the income, it is optimal, the optimal strategy of the incumbent is to accommodate. So what should the entrant do? If the entrant chooses in, then the incumbent will choose accommodate and he will get a payoff of 5. on the other hand if the incumbent chooses out he will get a payoff of 0 is 0 less than 5 yes. So, the entrant will choose in and the incumbent will choose accommodate. So, the entrant will choose in and the incumbent will choose accommodate and this constitutes what we call the sub game perfect Nash equilibrium.

Now why sub game perfect? Why such a name? Because if you take any sub game of this total game let us say this is a sub game, this is a sub game, right. If you take a sub game here also everybody is acting optimally, okay. given what has happened the incumbent is acting optimally in this sub game and in the total game which is the sub game of itself everybody is acting optimally, okay. Hence this is called the sub game perfect Nash equilibrium, okay. Let us try to solve the other problem which was there, this one, let us see.

If player 1 chooses L then what will happen? Well if player 1 chooses L, then player 2 can choose U or D and then player 1 can again act and choose L or R. We will always start at the bottom of the tree or we will start at the very end and then move back in time. So, which is the end of time here? This where player 1 is moving. So, given that L and U has happened, given that L and U has happened. and the player 1 is acting here what should he choose L or R remember the first payoff is that of player 1 if player 1 chooses L he gets 3 if he chooses R he gets 6 so what should player 1 choose the answer is player 1 should choose R he should choose R because then he gets a higher payoff which is 6 very good Now player 1 is choosing R and getting a payoff of 6 here.

Now if player 2, now we look at player 2 here, should player 2 when he is acting at this point, should he choose U or D? Let us see, if he chooses D his payoff is going to be if he chooses u then then if he chooses u then player 1 will choose r because that's optimal for player 1 to do and what will be player 2's payoff then 10 so what should player 2 do player 2 should choose u very good now what about player 1 standing here Let us do another thing. Let us consider player 2 here at this juncture. If player 2 arrives here, will he choose u or d? If he chooses d, his payoff is 10.

If he chooses u, his payoff is 2. So what should he choose? d. Great. Now look at player 1. Player 1 can take two paths. he can take two paths either he can take just a second let me use another color maybe, let me use blue. If he takes this path what will happen? If player 1 chooses L then player 2 will choose U and player 1 will again choose R and what will be player 1's final payoff? If player 1 chooses this direction, if he chooses R at the beginning, then what will happen? Then player 2 is going to choose D and player 1 gets a payoff of 5.

So choosing the left path gives him a payoff of 6, choosing the right path gives him a payoff of 5. So which route will player 1 take? Answer is simple. He is going to take the left route. So, this becomes my sub game perfect Nash equilibrium. Player 1 playing L, then player 2 playing U and then again player 1 playing R.

So, L player 1 playing L, player 2 playing U and player 1 again playing R, right. That constitutes my sub game perfect Nash equilibrium. Great I hope I have been able to convey the idea of a sub game perfect Nash equilibrium in a sequential move game. In the next lecture we will use this idea and we will introduce the idea of bar gaining. See you in the next lecture. Thank you.