Artificial Intelligence for Economics

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Lecture - 19

Lecture 19 : Game Theory (Contd.)

Welcome back to our next lecture in game theory. In the first lecture we have seen the idea of a Nash equilibrium and in a simultaneous move game what action profile or what action profiles constitute a Nash equilibrium or Nash equilibria. So now that we have familiarized ourselves with the idea of Nash equilibrium, right at the fag end of the last lecture we saw we encountered a game which was the penalty shootout game, where we saw that we did not have, we could not find a Nash equilibrium or a pure strategy Nash equilibrium. This is where we ended. but then we had another thought what was that ok we cannot compute or we cannot find a pure strategy Nash equilibrium for this game but what if we try to expand the idea of how we define strategy okay till now we have considered pure strategy that is a strategy of a player is simply an action which he is choosing in a simultaneous move game an action which he is choosing from his possible set of actions but what if he doesn't choose an action but he chooses different actions randomly What exactly do we mean by that? On that note, I will introduce the idea of mixed strategy.

So, what is a mixed strategy? Remember the setting of the game, do not lose sight of that picture. What do we have in a simultaneous move game? We have a bunch of players, we have a set of actions for every player. A player chooses one action from his possible set of actions such a couple of chosen actions constitute a action profile and any action profile can be a Nash equilibrium if given what a player is doing in that action profile or given what all the other players are doing in the action profile any particular player has no incentive to deviate from what he is doing. So that's where we were, just to give you a recap of where we stand right now.

So what is strategy chosen by a player? It is an action from his set of actions. For example, the striker's possible set of actions was to kick left or kick right. He can choose any one of those actions. That was his strategy, pure strategy. But now we are introducing something else.

We are talking about mixed strategies now. Now what is mixed strategy? Let us carefully define this. A mixed strategy of a player is nothing but a probability distribution over the set of actions of the player. So let us say a player has a set of actions given by the set A which is $\{a_1, a_2, \ldots, a_n\}$. So these are the possible actions which the player has.

Let us say a mixed strategy is a vector of probabilities $p_1, \ldots, p_n$ such that summation of $p_i = 1$ where p_i or p_j denotes the probability that action j has been action a_j has been chosen. To put it to put it crisply a mixed strategy is a probability distribution over my set of actions. So, right now I am not choosing an action from my set of possible actions instead what I am doing is. I am choosing a probability distribution $p_1, \ldots, p_n$ such that I choose action a_j with probability p_j.

So, this is a mixed strategy, great. So, now since we ended the last lecture by figuring out that this penalty shootout game does not have a pure strategy Nash equilibrium. Now we will inspect if this has a mixed strategy Nash equilibrium, okay. Let us see, let us understand that. So, the strikers action set is given by what? Remember KL and KR, kick to the left, kick to the right, we have seen it in the last lecture.

The goalkeepers action set is dive to the left and dive to the right, great. Now, what is a possible mixed strategy chosen by the striker? Well, it is a vector of probabilities p and $1-p$, where the striker chooses KL with probability P and KR with probability $1-p$. Similarly, the goalkeeper can also have a mixed strategy. He can choose dl with probability q and dr that is dive right with probability $1-q$. The question is if a striker chooses a mixed strategy of this kind p, $1-p$ and a goalkeeper chooses the mixed strategy q, $1-q$ can this constitute a Nash equilibrium okay.

And if so, for what values of p and q? That is what we will inspect in this lecture. Let's move on. By the way, before we move on, a little point. If p is 0, then what happens? If p is 0, then the striker is basically choosing kick left with probability 0. Remember p is the probability with which the striker is choosing KL.

If that probability is 0, it simply means that the striker is choosing a pure strategy KR. Similarly, if p is 1, that means the striker is choosing a pure strategy KL, okay. So, similarly for $q=0$ and $q=1$. Okay, let us move ahead. Let's understand it from the striker's perspective now.

So what is the goalkeeper playing? The goalkeeper is playing a mixed strategy q, $1-q$. That is, the goalkeeper is choosing DL, that is dive left with probability q and dive right with probability $1-q$. Okay? Very good. Now if the goalkeeper is choosing this mixed strategy, let us step into the shoes of the striker and analyze carefully. If the striker chooses KL, what is his expected payoff? Well, the goalkeeper is diving to the left with probability q.

Diving to the right with probability $1-q$. So the striker chooses KL that is kicked to the left then He will get a payoff of 0 if the goalkeeper is diving to the left and a payoff of v if the goalkeeper is diving to the right Correct remember the payoff matrix and So the expected payoff of the striker, the payoff matrix, by the way, you should refer to the previous lecture. The expected payoff of the striker when he's kicking to the left is given by what? It is $q \times 0+(1-q) \times v$, right? I repeat once more in case it's not clear. If the striker is kicking to the left if the striker is kicking to the left with probability q the goalkeeper is diving to the left and with probability $1-q$ the goalkeeper is diving to the right. If the goalkeeper dives to the right then the striker will score a goal but then he is right footed so when he is kicking to the left he gets a payoff of v.

right. So, with probability $1-q$ he will get v and with probability q the goalkeeper will stop the ball and he will get a payoff of 0. So, the expected payoff of the striker is simply $(1-q) \times v$. I hope this is clear. What happens if the striker chooses KR that is chooses to kick to the right. what will be his expected payoff well if the striker is choosing to the right remember the goalkeeper is choosing DL with probability q so if I am the striker and if I am kicking to the right and if the goalkeeper dives to the left I will score and my payoff is equal to 1 with probability $1-q$ the goalkeeper will dive to the right if I am also kicking right and if the goalkeeper is diving right my payoff is equal to 0.

So, with probability $1-q$ I will end up with a payoff of 0. So, what is my expected payoff $q \times 1+(1-q) \times 0$ that is the definition of expectation right or in other words it is q. So, now this striker Given that the goalkeeper is playing this mixed strategy $(q, 1-q)$, the striker will choose KL if $(1-q) \times v$ that is the expected payoff from kicking L is greater than the expected payoff this is my expected payoff from kicking left from KL, this is the expected payoff from choosing KR. So, the striker will choose KL when the expected payoff from choosing KL is greater than the expected payoff from choosing KR. If we rearrange the terms, we simply get this.

Similarly, if the striker, the striker will choose KR if the expected payoff from playing KL is less than the expected payoff from playing KR. rearrange the terms and we get this $q>\frac{v}{1+v}$, okay. Now, there is a problem, if the striker chooses KL that is if $q<v$ by $1+v$, if the striker chooses KL is it optimal for the goalkeeper to play this mixed strategy $(q, 1-q)$. The answer is no. If the striker chooses KL, then what is the goalkeeper's best response? We have seen it in the previous lecture.

It is to dive to the left, which means diving to the left is synonymous with choosing $q=1$, right. Similarly, if the striker chooses KR, What is the goalkeeper's best response? Will the goalkeeper keep playing a mixed strategy $(q, 1-q)$ with q being positive? The answer is no. The goalkeeper's best response is to simply choose the pure strategy DR, dive to the right. That is $q=0$. Now you see there is a clear mathematical contradiction which is taking place here.

If $q<\frac{v}{1+v}$, which is a fraction, so if q is a fraction, then the striker chooses KL, but if the striker chooses KL, the goalkeeper's best response is to choose $q=1$, which is not a fraction anymore. Similarly, so $q<\frac{v}{1+v}$ and $q=1$ these are contradictory. Similarly, if the striker chooses KR, sorry if $q>\frac{v}{1+v}$ the striker will choose KR but if the striker chooses kr the goalkeeper should choose DR which is synonymous with choosing q equal to 0 but $q>\frac{v}{1+v}$ and $q=0$ are in contradiction with each other okay so $q<\frac{v}{1+v}$ and $q>\frac{v}{1+v}$ leads to contradictions. In fact, what they actually do is they take us back to the pure strategy paradigm where we do not have a Nash equilibrium existing, okay. So, the goalkeeper choosing $q>\frac{v}{1+v}$ or $q<\frac{v}{1+v}$ both lead us back to the pure strategy paradigm where we did not have a Nash equilibrium.

So what do we do? What is the only other q left which we should focus on? Well, it is $q=\frac{v}{1+v}$, okay. So if $q<\frac{v}{1+v}$, the striker will choose KL. If $q>\frac{v}{1+v}$, the striker will choose KR. But if $q=\frac{v}{1+v}$, then what will happen? well then the striker is indifferent between choosing KL and KR okay which means now the striker will play a mixed strategy. In the previous scenario if $q\neq\frac{v}{1+v}$ the striker will not play the proper mixed strategy, the striker will actually play a pure strategy either KL or KR.

But if $q=\frac{v}{1+v}$, now the striker is indifferent between choosing KL and KR and he can actually randomize between KL and KR, because he gets equal payoff by choosing KL and KR if $q=\frac{v}{1+v}$. And what is the strikers expected payoff? Let us say he plays a mixed strategy $(p, 1-p)$ that is he chooses KL with probability p and KR with probability $1-p$ then what happens? Well, if he chooses KL what is his expected payoff? $1-q\times v$. If he chooses KR what is his expected payoff? q. Now, he is choosing KL with probability p.

So, this is the payoff. This is the expected payoff I am sorry, this is the expected payoff from choosing KL, expected payoff from choosing KL. This is the expected payoff from choosing KR this is KL KR and the striker is choosing KL with probability p and KR with probability $1-p$. So, what is the strikers expected payoff? Well it is p into the expected payoff from KL plus $1-p$ into the expected payoff from KR. and it turns out to be $\frac{v}{1+v}$ which is quite obvious great. So, no matter what p chooses the strikers expected payoff is $\frac{v}{1+v}$.

So, no matter what p the striker chooses. So, then the striker can actually choose a $p>0$ because he is indifferent between choosing p for he is indifferent between choosing any p actually. Now, let us look at the goal keepers perspective. Given that the striker plays KL and KR with probability p and $1-p$, let us now step into the shoes of the goal keeper and let us see how he is thinking, let us think with him. the striker is playing this mixed strategy $(p, 1-p)$ that is he is kicking to the left with probability p and kicking to the right with probability $1-p$.

So, if the goalkeeper chooses to dive to the left what will be the payoff? Well with probability P the goal the striker will also kick to the left which means with probability P the goal will be stopped and the payoff 0 for the goalkeeper and with probability $1-p$ the striker will kick to the right and if the goalkeeper is diving to the left a goal will be scored. So, the goalkeepers payoff is going to be -1. So, what is the expected payoff from playing DL or what is the expected payoff of the goalkeeper if he dives to the left it is simply minus of $1-p$. Very similarly what is the goalkeepers payoff if he dives to the right. is very simple with probability p the striker will kick to the left which means a goal will be scored with a v probability.

So, this is the payoff if the striker kicks to the left this is the payoff if the striker kicks to the right remember the goalie is diving to the right. If the goalkeeper is diving to the right and the striker also dives to the right, his payoff is 0. The goalkeeper's payoff is 0 because the goal is stopped. If the striker dives to the left, the goalkeeper's payoff is $-v$. Remember the payoff matrix from the previous lecture.

In case you are confused, please go back to the previous lecture and take a look at this particular payoff matrix. So this is the expected payoff of the goalkeeper if he chooses to dive to the right, great. Now what will the goalkeeper choose? Exactly like the analysis of the previous slide where we looked at the strikers perspective, what should the goalkeeper do now? Well the goalkeeper will choose DL. If the expected payoff from choosing dl is greater than the expected payoff from choosing dr right, which is given by this is the expected payoff from choosing DL, this is the expected payoff from choosing dr. So, if the expected payoff from choosing DL is higher, the goalkeeper will choose dl.

If the expected payoff from choosing dl is lower than the expected payoff from choosing dr, the goalkeeper will choose DR. Now, if we rearrange the first in equation we get this and if we rearrange the second in equation we get this. So, the goalkeeper will choose dl if $p > \frac{1}{1+v}$ and the goalkeeper will choose DR if $p < \frac{1}{1+v}$. Now, we are again stuck in a very similar problem which we encountered 5 minutes before. If the goalkeeper chooses DL, that is if the goalkeeper chooses to dive to the right, then what is the striker's best response? Well, then the striker's best response is to kick to the right.

If the goalkeeper chooses DL, that is dive to the left, the striker's best response clearly is to kick to the right. But kicking to the right actually means choosing $p=0$. Remember what is the, what is the mixed strategy the striker is playing? He is choosing KL with p and KR with $1-p$. So playing KR with certainty means choosing $1-p$ equal to 1, that is $p=0$, right. similarly if the goalkeeper chooses dr that is if the goalkeeper chooses to dive to the right which is what he does when $p < \frac{1}{1+v}$ then what will the striker do if the goalkeeper dives to the right the striker will necessarily choose KL kick to the left but choosing KL with certainty simply means it is synonymous with choosing $p=1$ Okay, now we again arrive at contradictions or we are encountering the same problems which we did before, just 5 minutes before.

Remember, if the goalkeeper is choosing dl, when is the goalkeeper choosing dl? When $p > \frac{1}{1+v}$. But if $p > \frac{1}{1+v}$, the goalkeeper chooses dl which in turn incentivizes the striker to choose KR that is $p=0$. But $p > \frac{1}{1+v}$ and $p=0$ they do not go they are contradictory. Similarly, $p < \frac{1}{1+v}$ and $p=1$ are contradictory. And more importantly what we what this leads to is it leads back to the pure strategy paradigm which is what we want to avoid because we encountered that there does not exist an ashe equilibrium ok, in a pure strategy setting.

So, there is no pure strategy Nash equilibrium of this scheme, but if $p \neq \frac{1}{1+v}$ that is where we are led back to. So, what is the only other p which is worth investigating? Well, it is $p = \frac{1}{1+v}$, ok. Let us see what that is, what happens when the striker chooses this mixed strategy where $p=\frac{1}{1+v}$ well then the goalkeeper is indifferent between choosing dl and dr right remember if $p>\frac{1}{1+v}$ the goalkeeper chooses DL if it is less than $\frac{1}{1+v}$ the goalkeeper chooses DR if it is equal to $\frac{1}{1+v}$ well the goalkeeper is indifferent between choosing DL and DR right. So, in that case the goalkeeper can actually randomize and choose any mixed strategy $(q,1-q)$ with a positive q and then what will be the expected payoff of the goalkeeper? Well, it will simply simplify to $\frac{-v}{1+v}$ which is independent of q which means if the striker chooses this mixed strategy where $p=\frac{1}{1+v}$ then the goalkeeper, the goalkeeper's payoff is equal to $\frac{-v}{1+v}$ no matter what mixed strategy he chooses, okay. So, what is the final solution? So, when will, when will the striker randomize? Or when is when will the goalkeeper randomize? The goalkeeper will choose or randomize between dl and dr with a positive q only if $p=\frac{1}{1+v}$.

If p is not equal to $\frac{1}{1+v}$ as we see in this slide the goalkeeper will necessarily choose a pure strategy either dl or dr. But if $p=\frac{1}{1+v}$ the goalkeeper can choose an actual a proper mixed strategy with a positive q also when will the striker randomize or when will the striker play a proper mixed strategy with a positive p that is when $q=\frac{v}{1+v}$ because remember in this slide we have mentioned that if $q\neq\frac{v}{1+v}$ the striker will choose a pure strategy either KL or KR So, the striker will have an incentive or the striker will actually choose a proper mixed strategy with a positive p only if $q=\frac{v}{1+v}$, ok. Also, also if $p=\frac{1}{1+v}$, the goalkeeper can choose any mixed strategy and his payoff is going to be the same, right. if the goalkeeper chooses $q=\frac{v}{1+v}$ the striker can choose any mixed strategy any p and his payoff is going to be the same which means that the striker can choose that particular $p=\frac{1}{1+v}$ ok. So, if we look at the first 4 points what can we infer from that we infer the following if the striker chooses $p=\frac{1}{1+v}$ and the goalkeeper chooses $q=\frac{v}{1+v}$, they work.

Why do they work? Because of the following reason. If $p=\frac{1}{1+v}$, that's the only p chosen by the striker for which the goalkeeper will play a pure mixed strategy. And any mixed strategy which the goalkeeper plays yields him the same payoff. that is and hence if the goalkeeper plays this mixed strategy with $q=\frac{v}{1+v}$, he will get the same payoff compared to what had he chosen a different q, okay. So, he cannot do any better. So, he cannot do any better remember the idea of Nash equilibrium given a strategy chosen by a player.

the other player should not have any incentive to deviate from what he is doing, right. So, if the striker chooses a mixed strategy with $p=\frac{1}{1+v}$, no matter what q the goalkeeper chooses his payoff will be the same. So, if he chooses $q=\frac{v}{1+v}$, choosing any other q will not yield any more payoff to the goalkeeper. Similarly, if the goalkeeper chooses $q=\frac{v}{1+v}$, the mixed strategy $(q, 1-q)$ with $q=\frac{v}{1+v}$, then the striker has no incentive to deviate and choose any other p, because his expected payoff is going to be equal for any p he chooses, that is what we have seen, ok. So, these two mixed strategies constitute a mixed strategy Nash equilibrium.

In the next class, we look at, we will relax the assumption of complete information. We will also look at an example very similar to this and we will solve it. We will solve for a mixed strategy Nash equilibrium. That will hammer in the idea even more into your head when you watch the next video. and we'll gradually step into the paradigm of incomplete information that's it for this lecture see you in the next thank you