Selected Topics in Algorithm

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Lecture No # 08
Module No # 02
Augmenting Path

Welcome so in the last lecture we have seen Kargar's Min cut algorithm and in this lecture we will see another interesting problem called matching in non bipartite graph.

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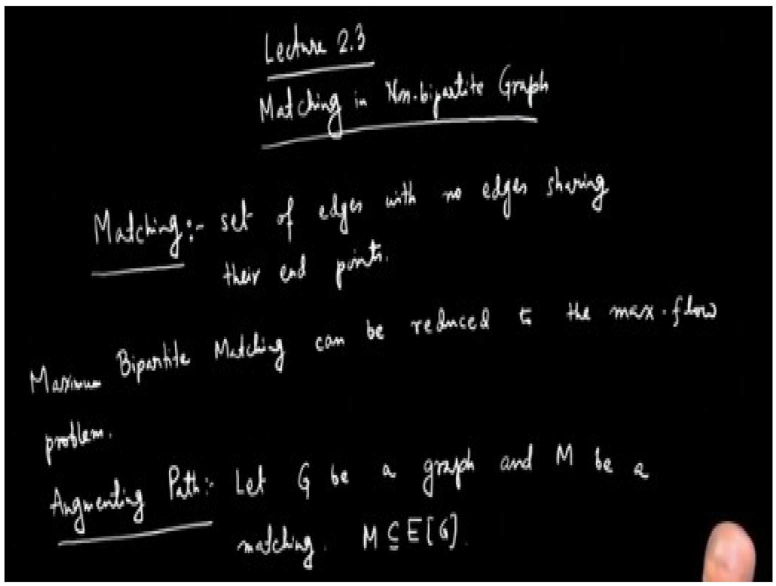

So lecture let us recall what is the matching? So matching is a set of edges with no edges sharing their end points. And for bipartite graph a bipartite maximum matching the maximum bipartite matching problem is where we need to find a maximum matching in a bipartite graph this can be reduce to flow problem to the max flow problem. So now in this lecture we are talking about matching in a non-bipartite graph or general graph and for that we need to build some concepts.

So there is a concept called augmenting graph, so augmenting path so what is an augmenting path? So let G be a graph and M be matching. Let us recall M is a subset of each set of G the subset of edges.

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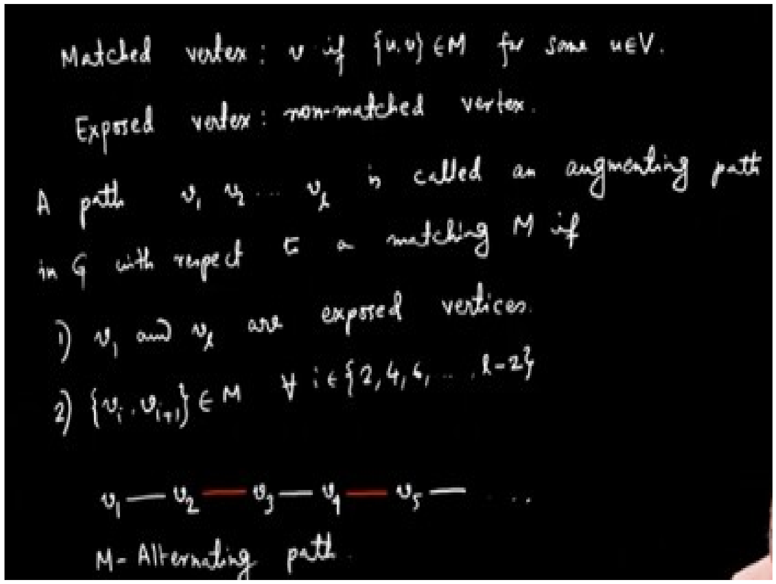

So there are 2 set of vertices one set of vertices are called matched vertex is v such that there exist an edge such that or let me write this way v is a matched vertex. If u, v belongs to matching for some vertex u belongs to v. The other vertices are called exposed vertices these are non-matched vertex. And what is an augmenting path? A path $v_1, v_2, \ldots,$ is called an augmenting path in G with respect to a matching M, if first condition is the first and last vertex must be exposed $v \in V_l$.

v_1 and v_2 are exposed vertices, then the alternating edges will be matching edges so v_i, v_{i+1} belongs to M for all $i \in \{2,4,6,\ldots,l-2\}$ because v_{l-1} is exposed to v_{l-1} will be matched. So these are so how does this path look like so here is v_1 there is a normal edge non matching edge v_2. And there is a next edge is a matching edge v_3 and next edge is normal edge v_4 and next edge is matching edges v_5 normal edge and so on.

This is how it looks like and if we drop this condition 1 then this path this is called alternating path also M alternating path. In this path edges from the matching and out of the matching are alternative.

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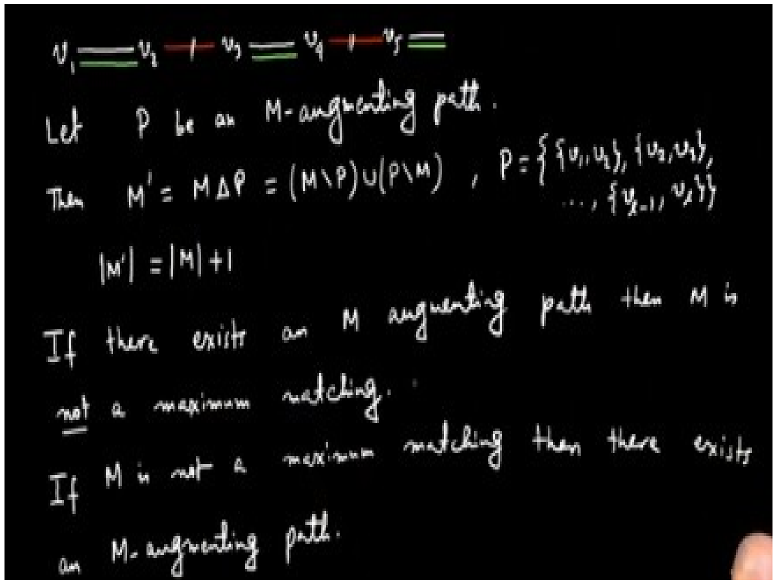

Now why this augmenting paths are useful we observe that if I have an augmenting path say suppose v_1, v_2 I have an augmenting path v_4, v_5 and so on. These are non-matching edges and these are matching edges. So if I from M if I drop this matching edges in this path and in an alternatively pick the non-matching edges in the matching. Then because the first vertex and last vertex they are exposed this particular operation increases the size of the matching.

So let p b and M augmenting path, then $M^{'}$ defined as M symmetric difference p, this is $(M \setminus p) \cup (p \setminus M)$. Think of p as the set of edges in the path so p is the set of edges in the path $v_1, v_2, v_3, \ldots v_{l-1}, v_l$ this v. And we observe that you know size of $M^{'}$ even this because of this exchange in this path as increased by 1 from the size of M. So whenever I have a matching M and I have an M augmenting path I can use this use this M augmenting path to increase the size of the current matching M.

So in particular if they are exists and M augmenting path then M is not, maximum matching. Interestingly the converse is also true, so if M is not maximum matching then they are must exist and him augmenting path so this was done.

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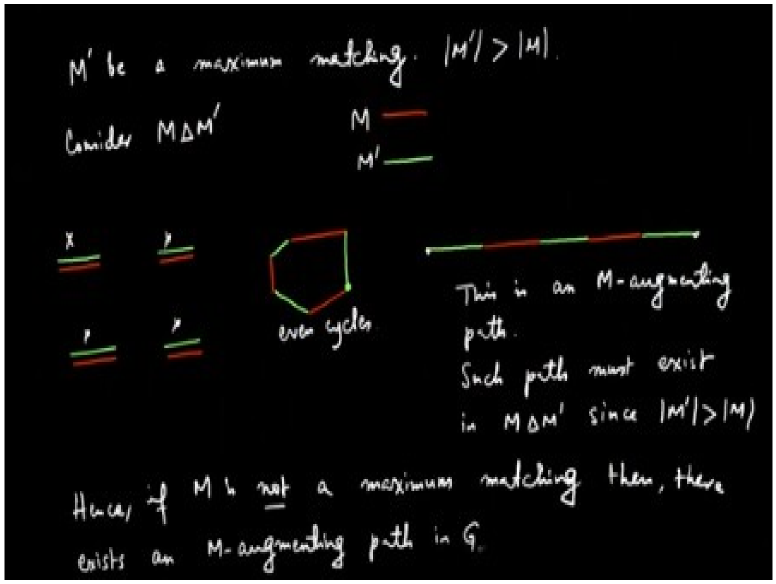

So let us prove this part the second part so suppose M is not maximum matching, and $M^{'}$ be maximum matching. That is $M^{'}$ size of $M^{'}$ is strictly more than size of M simply because $M^{'}$ is maximum matching and M is not a maximum matching. So let us consider the symmetric difference m symmetric difference $M^{'}$ how does it look? Like there will be edges.

So let us denote M with matching edges with under M with red color and matched edges under matching edges under $M^{'}$ with green color. So there will be edges which are both belong to M and $M^{'}$ these are the common edges. So they will get canceled in m symmetric difference $M^{'}$ so this thing will not be there and you know there will be edges. Let us take a so this sort of edges which belongs to both M and $M^{'}$ they will not belong to the symmetric difference.

So let us see how other edges look like so let us take a red edge matching from M so let us asks its end points is this endpoint is matched under $M^{'}$. If it is matched this looks like this, and let us ask where this end point; this end point is matched under red if it is yes then you continue here and continue like this. So m symmetric difference $M^{'}$ will have this, even cycles so this will be even cycles.

These cycles cannot be odd cycle because this, matching edges are alternating. Now in this, even cycles same number of edges from M and $M^{'}$ participant. So the edges there are some edges belong to both M and $M^{'}$ they contribute same to the, to both M and $M^{'}$ these even circuits also contribute same. Next but there are more green edges than red edge so there is at least, so there will be a paths like this that you know there is a green edge and this end point is matched under red also.

So if there is a path like this that means which starts with green edge and ends with green edge. So it can be long also but it has to start at green edge and end at green edge then this is and M augmenting path. If this end vertices are exposed under M and this has to be the case so this sort of paths such path must exist in M symmetric difference M'.

Simply because size of M' is strictly more than size of M if such paths are not there if it all you know even cycles and this edges this common matched edges. Then a size of M is same as size of M' and in that case you know M is also a maximum matching which will be a contradiction. So we have proved that you know if M is a, is not a maximum matching; hence if M is not maximum matching then they are exists and M augmenting path is G.

So these will be important construct of our algorithm important tool offer algorithm. So in some sense you know there could be a hypothetical algorithm like start with a then edge and make it a matching singleton matching and if there is an augment look for aim augmenting paths. If there is an augmenting path increase the size of the matching along that path.

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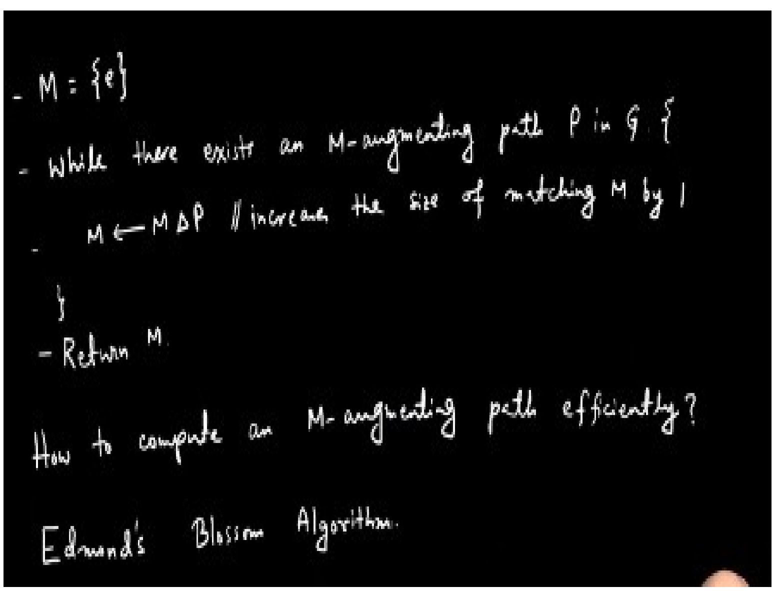

And so let us see so initialize M to be any edge m and while there exists and M augmenting path in g augmenting path $p \in G$. And update M is M symmetric difference p this increases the size of matching M by 1. And then return M so it all boils down to how to compute an M augmenting path efficiently, so how to compute and M augmenting path efficiently. And here comes the beauty of Edmond's algorithm it is called Edmond's blossom algorithm.

So Edmond blossom algorithms solved this problem of finding and computing an aim augmenting path efficiently using a brilliant idea that we will see next. But before that we need some lower bound on we need some bound or characterization of size of maximum matching.

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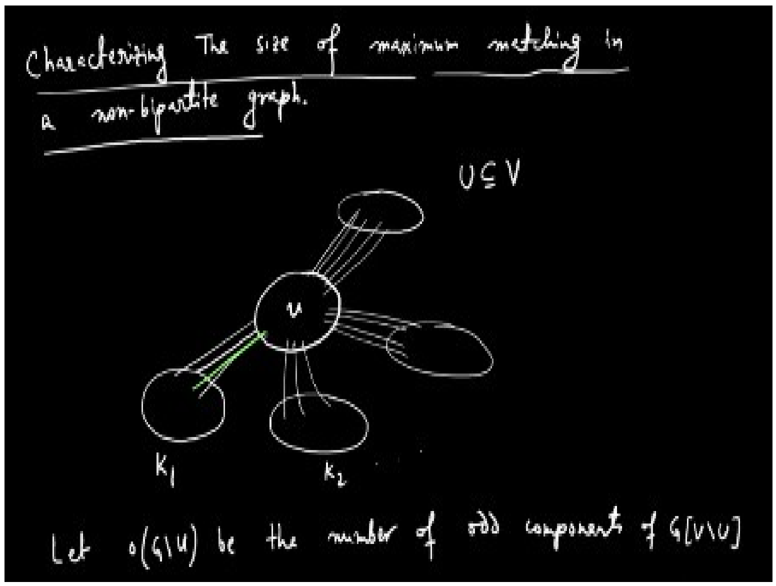

So character rising the size of maximum matching in a non-bipartite graph. So let G be a graph and let you be a subset of u is a subset of v and cardinality of u is odd u contains an odd number of vertices, it is not good u be any set of vertices. And now we ask if we delete the vertices of the cardinality of u could be anything if we delete the set of vertices u from the graph we will get various kinetic components these are some edges are here.

Now we ask how many of these components suppose these are his k_1 this is k_2 and so on. So if on if a component has size even that means if any component has even number of vertices the matching edges and could they could be completely matched inside. But if a vertex has an odd number of vertices then all the vertices cannot be matched inside and at least so suppose k_1 has this size is odd then for maximum matching or to match all these vertices you know at least one of this cross edge must be used.

So what could be the size of matching that this vertex u or this set of set u imposes. So let $o(G\backslash u)$ be the set of be the number of odd components of $G[V\backslash u]$. Now you see that all odd components need one vertex from u to get matched but u has only cardinality u many vertices.

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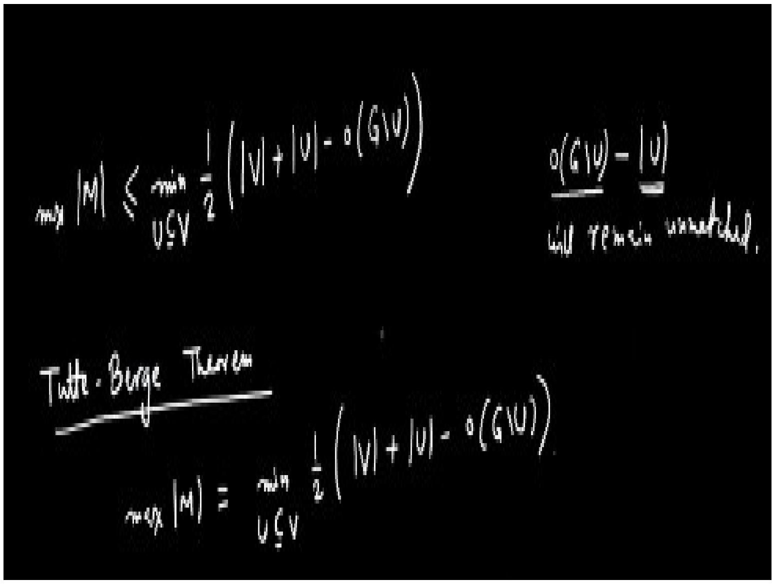

So $o(G\backslash u)$ is the number of odd components and u can supply at most cardinality of too many vertices cardinality of as its partner, so this many vertices so $o(G\backslash u)$ this will remain unmatched. Because each of the odd components of $G\backslash u$ needs a partner but in u there are only cardinality of human partners. So this many $o(G\backslash u)$ this many components will not get a corresponding partners from u and at max of everything else may be matched.

So the size of the matching could be at most this and if each such subset u places this upper bound. So if I take minimize this quantity then this is of course places a upper bound on maximum matching. So cardinality of m maximum matching and what Tutte-Berge formula is this inequalities indeed Tutte-Berge theorem says that max M is min u substitute of v half of cardinality v + cardinality u - number of odd components in $G\backslash u$.

So this will be used in proving the optimality of the matching found in Edmonds Blossom algorithm. We will show an exhibitor of a set of vertices u and show that the matching output is indeed this quantity which will also prove the Tutte-Berge theorem because this quantity is an upper bound. So we will continue in the next lecture.