Selected Topics in Algorithm
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Lecture - 56
Primal Dual Scheme

Welcome, so in the last class we have seen the randomized rounding algorithm for set cover and the two-factor approximation algorithm for vertex cover. In this lecture we will see another technique of designing combinatorial algorithm, combinatorial approximation algorithm using the framework of linear programming duality and that is called LP duality schema, it is called primal dual schema.

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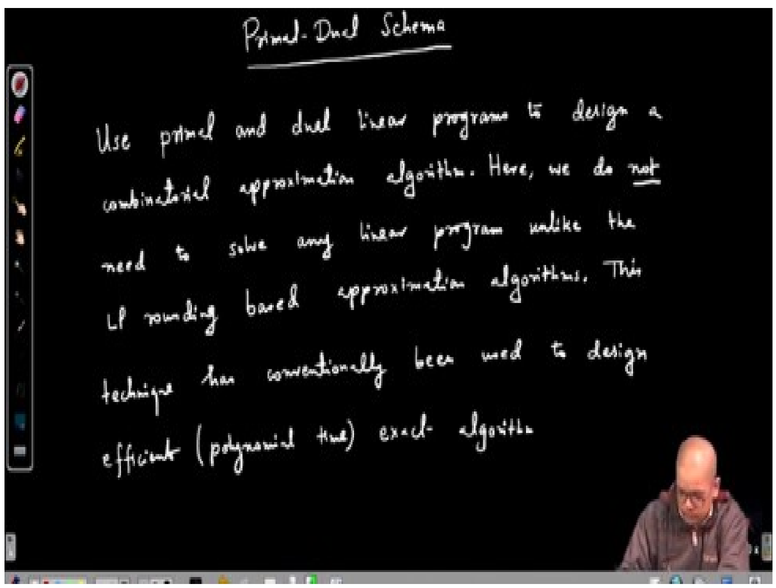

Recall you know we have discussed that there are two ways to desire to use linear programming for approximation algorithm design. One is directly solve the linear program and use the solution to come up with an approximation algorithm. These are like linear programming rounding techniques. The other is use linear program to either design an approximation algorithm or to analyse it.

Now we have seen the method of dual fitting to analyse a combinatorial algorithm for combinatorial approximation algorithm using linear programming duality. Primal dual schema is another technique of for designing a combinatorial algorithm based on linear programming duality combinatorial approximation algorithms. So, again in this method we are not solving the linear program.

So, idea is use primal and dual linear programs to design combinatorial approximation algorithm. Here we do not need to solve any linear program unlike the LP rounding based approximation algorithm and this primal dual schema has been used you know more effectively for designing exact algorithm. So, this schema this technique has conventionally been used to design efficient in algorithm design by efficient polynomial time. If not mentioned otherwise it means polynomial time efficient exact algorithm.

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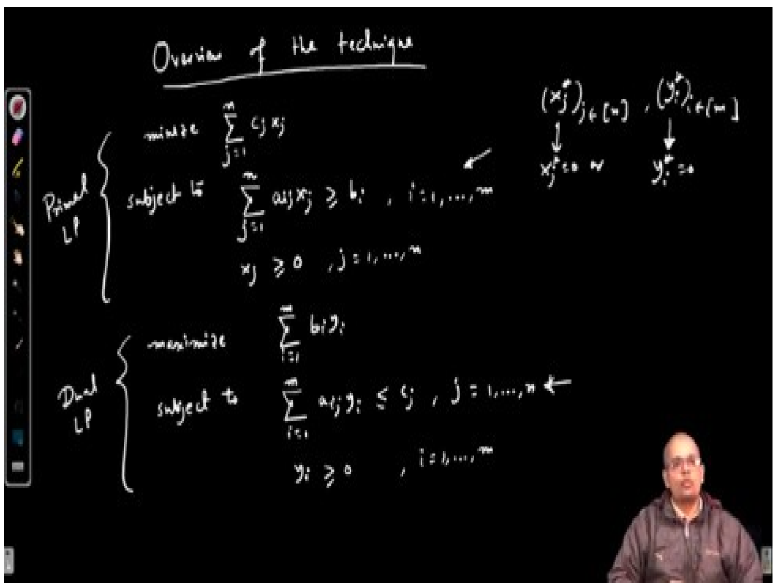

So, we suitably extend this technique to design efficient approximation algorithm. So, what is the technique so, overview of the technique? Suppose we write our problem as a linear program in a standard form in integral linear program and then we relax it to a linear program and we have written it in a standard form that means minimize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \ge b_i$.

This is for i = 1 to m we have m such constraints and x_j is greater than equal to 0 for j = 1 to n. So, this is a linear program in standard form and let us call it the primer linear program, this is the primal. Now let us write this dual linear program which is maximize $\sum_{i=1}^{m} b_i y_i$ subject to

$\sum a_{ij} y_i \leq c_j$, i = 1 to m. This is we have for all j = 1 to n and then we have $y_i \geq 0$ for i = 1 to m this is the dual linear program.

Now recall we have seen the complementary slackness condition it is a constraint it is an equivalent condition when a solution for primal and a solution for dual when it is optimal. Let us recall so suppose $(x_j^*)_{j\in[n]}$ and $(y_i^*)_{i\in[m]}$ you know these are one is primal solution another is dual solution and both are optimal if and only if whenever you know either x_j^* a 0 or if it is non zero then the jth constraint jth dual constraint is type that constraint that inequality holds with equality.

That is a primal complementary slackness condition and same is for dual. It means this dual solution is optimal even if and only if y_i^* if it is 0 for all $i\in[m]$ or the ith constraint ith primal constraint is tight it holds with equality now that is for exact solution. But now we are what we are looking for over? Looking for approximate solution. So, we extend this complementary slackness condition for the need of designing approximation algorithm.

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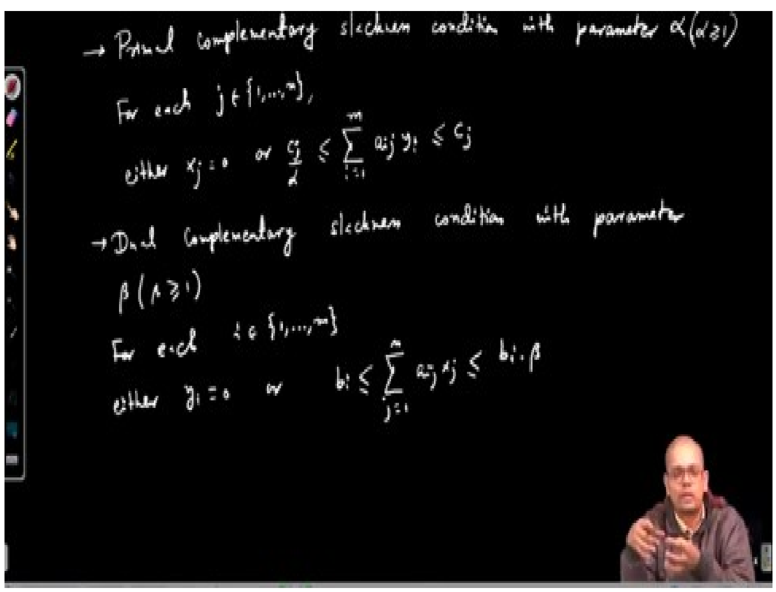

So, let me write. So, let me state two conditions primal complementary slackness conditions with parameter alpha. What is the condition? That alpha of course alpha is greater than equal to 1 that for each primal variable are $x_1,\dots,x_n$ for each $j\in[n]$ either x_j is 0 or the dual constraint the jth dual constraint hold approximately. It does not hold with exact equality but it should not approximately calling.

What do I mean by that jth dual constraint? It is of course needs to be less than equal to c_j but it is at least $\frac{c_j}{\alpha}$ that is what we mean by it holds with equality approximately. It is not it is off from equal to c_j by at most of factor of α it is of from equality by at most of factor of α. So, this is primal complementary slackness condition with parameter α.

Similarly, let us define dual complementary slackness condition with parameter β. Of course, α is also greater than equal to 1. What is the constraint again? For each dual variable you know $i \in [m]$ either $y_i = 0$ or the ith primal constraint that means $\sum_{j=1}^{n} a_{ij} x_j$ this is of course greater than equal to b_i. So, this will hold this greater than equal to b_i but this is less than equal to b_i times β.

So, now what will happen if a solution a primal solution and a dual solution need not be optimal satisfies this approximate version of complementary slackness conditions.

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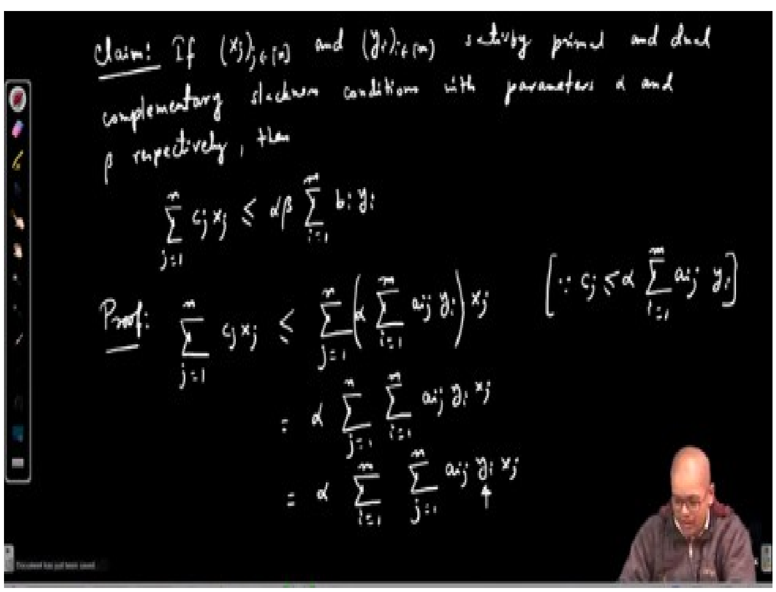

Then we claim that now we will see that these solutions will help us to design an approximation algorithm. So, for that is right claim so these are in some sense x and y because they are satisfying the primal and dual complementary slackness conditions approximately. So, they are approximately optimal that is the intuitive statement of this claim. So, if $x_j, j \in [n]$ and $y_i, i \in [m]$ satisfy primal and dual complementary slackness conditions with parameters α and β respectively.

Then we have the cost of the objective function primer objective function at this primal solution is approximately minimum it is not too high that is the claim that $\sum_{j=1}^{n} c_j x_j$ this is less than equal to $\alpha \beta$ times summation. You know in the exact complementary slackness condition the primal solution and dual solution these values will coincide but here it will coincide approximately with the factor of $\alpha \beta$.

So, what is the dual solution is the value of the dual objective is $\sum_{i=1}^{m} b_i y_i$. Proof, $\sum_{j=1}^{n} c_j x_j$. Now we replace c_j so c_j is at you I will use this inequality figure is at most α times $\sum_{i=1}^{m} a_{ij} y_i$. This is at most $\alpha a_{ij} y_i x_j$. So, that is what we do. We simply use that since c_j is less than equal to $\alpha \sum_{i=1}^{m} a_{ij} y_i$.

Now what we do is that we take α outside this is $\alpha \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} y_i x_j$. Now when we have two finite sums finite double sums when we can reverse the order of the sum that we can do if both the sums are finite. So, this is $\alpha \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_i x_j$. Now y_i does not depend on j so we take y_i outside of the inner sum.

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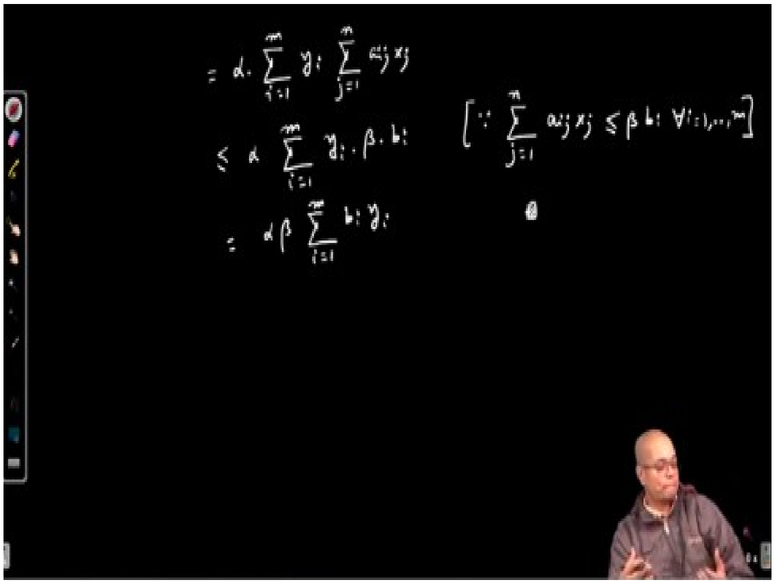

So, this is $\alpha \sum_{i=1}^{m} y_i \sum_{j=1}^{n} a_{ij} x_j \leq \alpha \sum_{i=1}^{m} y_i \beta b_i = \alpha \beta \sum_{i=1}^{m} y_i b_i$.

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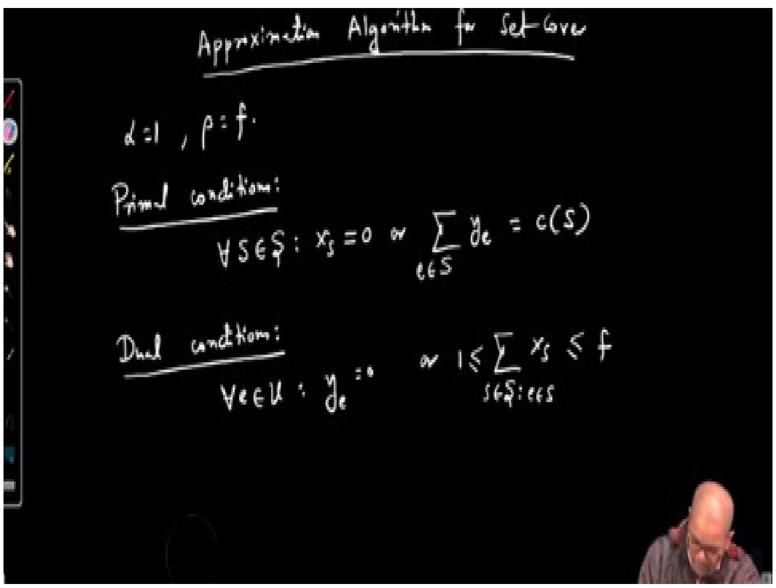

Now we will see using primal dual schema we can design a set cover approximation algorithm for set cover. So, approximation algorithm for set cover. The idea is to come up with primal and dual solutions which satisfy the complementary slackness conditions with parameter α and β. So, we will choose $\alpha=1$ and $\beta=f$ was the maximum frequency of an element. It is the maximum number of sets that an element belong to.

And this is the typical in designing approximation algorithms using primal dual schema. We typically set either $\alpha=1$ and β = the target approximation ratio or β = 1 and α = target approximation ratio. So, what are the primal conditions let us write. For set cover instead LP the primal conditions for all set is in this either x is 0 or summation because $\alpha=1$ primal conditions has to be satisfied exactly.

Summation $e\in U$ such that no summation e belongs to s no set is over packed, y_e = cost of s. So, this is the primal condition primal complementary selectness condition with $\alpha=1$ and the dual conditions. This condition this is a primary condition and dual condition is for all element in u we have either $y_e=0$ or you know dual condition is satisfied approximately. That summation the constraint the primal constraint corresponding to e which is $\sum_{s\in S:e\in s} x_s \leq f$ greater than equal to 1 because β is f.

Now what we will do is that we will change these variables x and y's. These solutions will change in such a way that you know this dual feasibility is ensured. So, here is a natural algorithm increment primal variables until some dual condition becomes tight. So, do not this just let me just clearly explain the algorithm and then we will see.

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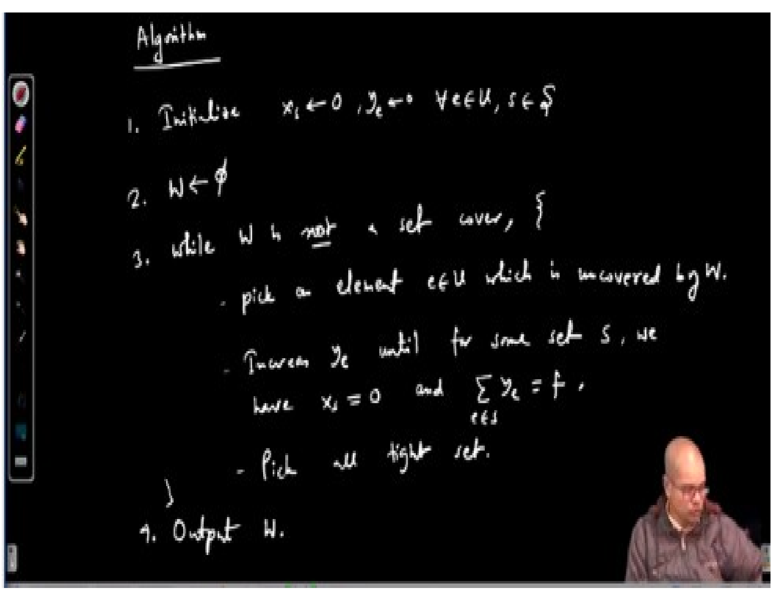

So, let us see the algorithm. So, initialize you know all x_s to 0 and y_e to 0 this is for all $e\in U$ and $s\in S$. So, this ensures that at the beginning you know all these two conditions primal condition and dual condition are stress satisfied because all variables are set to 0. Now what we do is that we maintain we built a set cover incrementally. So, while W is not a set cover what we do? Pick an element $e \in U$ which is uncovered by W.

Now what we do is that for that variable which is maybe now at 0 we increasing its value unless the if we increase the value, you know this primal condition needs to be satisfied. So, we keep increasing the value unless we keep increasing the value of y as long as we can and when some set. So, if we keep increasing this equality this summation y this dual constraint may become tight and then what we do is that we pick that set and for that x_s becomes 1.

So, let me write. Increase y_e until for some set s we have s is not equal to 0 sorry S = 0 it is not picked and summation y_e in s becomes tight because further y cannot be increased. Now what we do is we pick such a set. Then pick so these sets are called tight sets. So, pick all tight sets. What is a tight set? As it is tight is if summation y_e in is equal to f so that is the while loop, 4 is output W.

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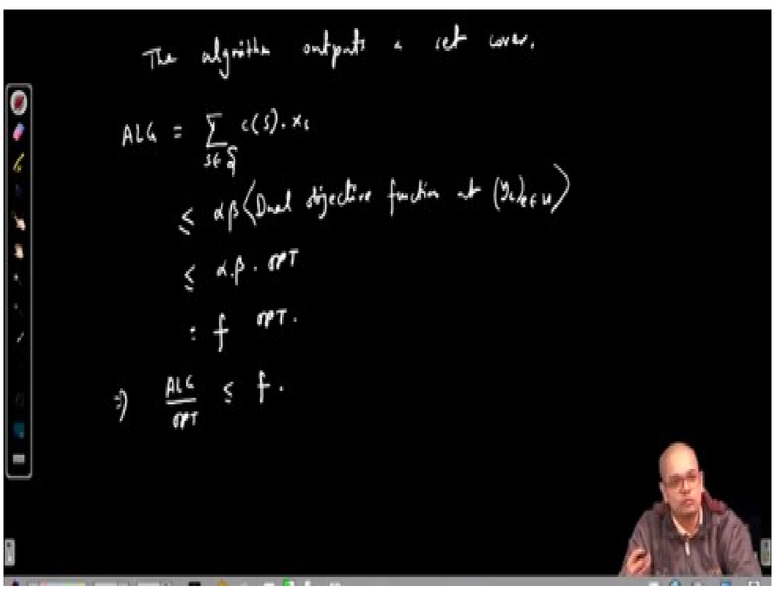

Now you know the analysis is trivial because the algorithm by design output are set cover always. Now what is the approximation ratio? What is ALG? ALG is summation and also observe that you know in every iteration the primal and dual complementary slackness conditions are satisfied. So, these conditions both the conditions are satisfied throughout the run of the algorithm.

It is satisfied at the beginning and throughout the run of the algorithm it is satisfied. So, ALG gives $\sum_{s \in S} c(s) x_s$ and but this is what? This is at most alpha times beta times dual. Let us write this way because the dual objective function is not essential is the dual objective function at y_e, e in a and the dual objective function value is a lower bound on OPT. This is alpha times beta times OPT.

So, the approximation ratio is alpha times beta and we have picked $\alpha = 1$ and $\beta = f$ this is f times of OPT. So, hence ALG by OPT is less than equal to f. Hence the approximation factor is at most f so, using the schema whenever we design an approximation algorithm the approximation factor is $\alpha\,\beta$. So, to design a better approximation algorithm we should try to minimize the factor $\alpha\,\beta$. So, let us stop here today.