Selected Topics in Algorithms
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Module No # 11
Lecture No # 54
Vertex Cover LP

Thank you welcome so in the last class we have looked at how we can have a 2 factor approximation algorithm using if factor approximation algorithm for set cover. In this class we will have a direct approach for having a 2 factor approximation algorithm for say for vertex cover.

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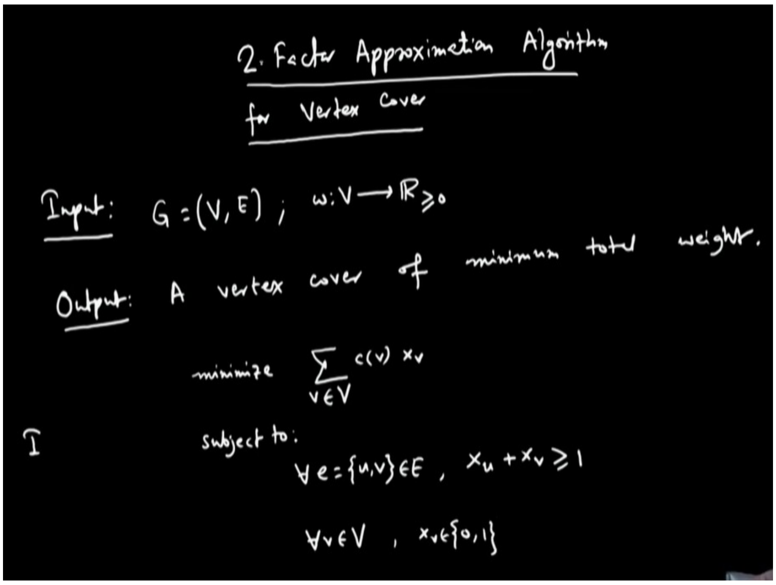

So 2 factor approximation algorithm for vertex covered then we can solve the general weighted version. So, each vertex what is the input a graph $G=(V,E)$ and there is a weight function from V to real numbers non-negative weights and what is the output? Vertex cover of minimum total weight so let us write the LP for vertex cover first right ILP minimize for each vertex I have a variable.

So $c(v)x_v$ this variable takes value 1 if we pick the vertex cover otherwise it takes value 0 this is v in v subject to the constraints we have a constraint for each edge. So for each edge $e=\{u,v\}\in E$ what are the constraint? We have $x_u+x_v\geq 1$ and for each vertex $v\in V$, x_v this is greater than this is in between in the set 0 and 1 so this is the ILP.

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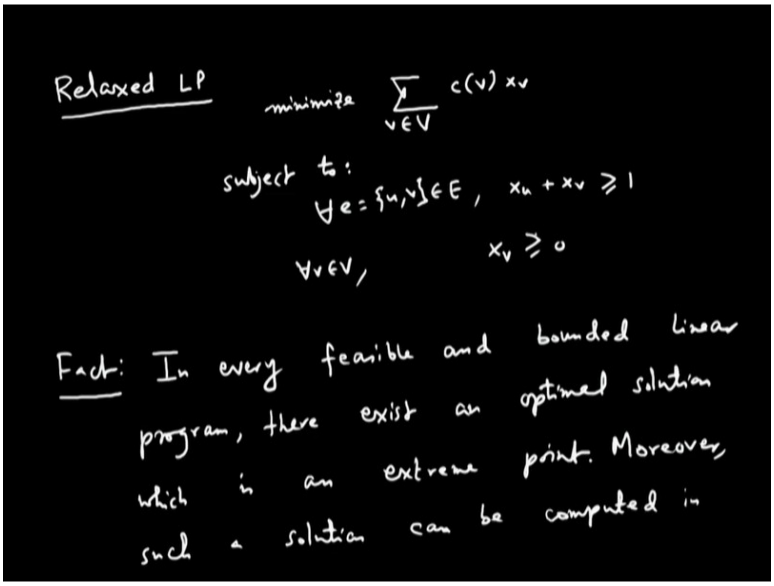

So what is the LP relaxation? Minimize $\sum_{v \in V} c(v) x_v$ subject to for each edge $e = \{u, v\} \in E$ $x_u + x_v \geq 1$ and x for every vertex $v \in V$ x_v is in between is greater than equal to 0 and less than equal to 1. But again because we are minimizing it and the objective function is monotonically increasing with x_v we can drop the condition that x_v less than equal to 1 without changing the optimal value.

Now we claim that this optimal solution and, an x so it follows from so there is a fact there in every feasible and bounded that means where the maximum or minimum is not infinity or minus infinity. In every feasible and bounded linear program there exist an optimal solution which is an extreme point.

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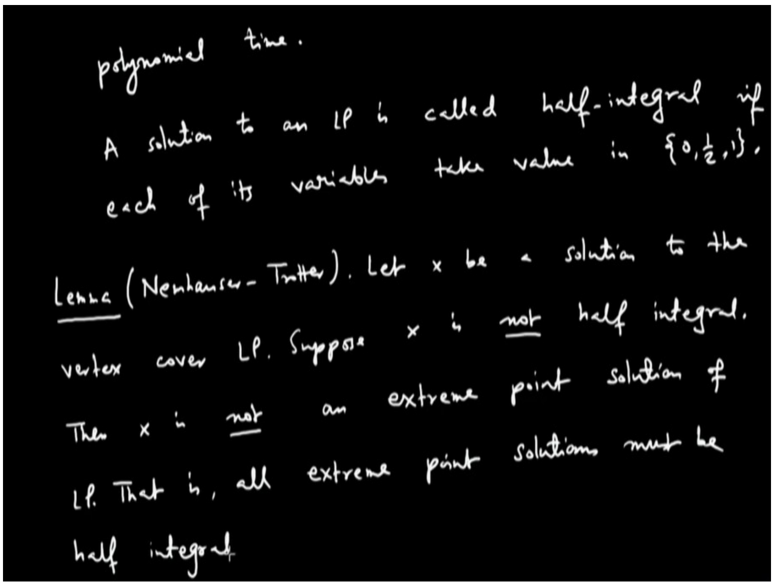

Moreover such a solution can be computed in polynomial time to an LP is called half integral if each of its variables take value in 0 half 1. So the claim is that for vertex covered the LP has an extreme point actually all extreme point solutions of the vertex cover all extreme points are half integral in particular the extreme point or those extreme points which are optimal solutions they are also half integral.

So here is the lemma this is called, Nemhauser Trotter theorem concerns optimal solution to the vertex cover LP. Suppose x is not half integral then x is not an extreme point solution of LP that means all extreme point solutions of the LP must be half integral that is on extreme point solutions must be half integral proof.

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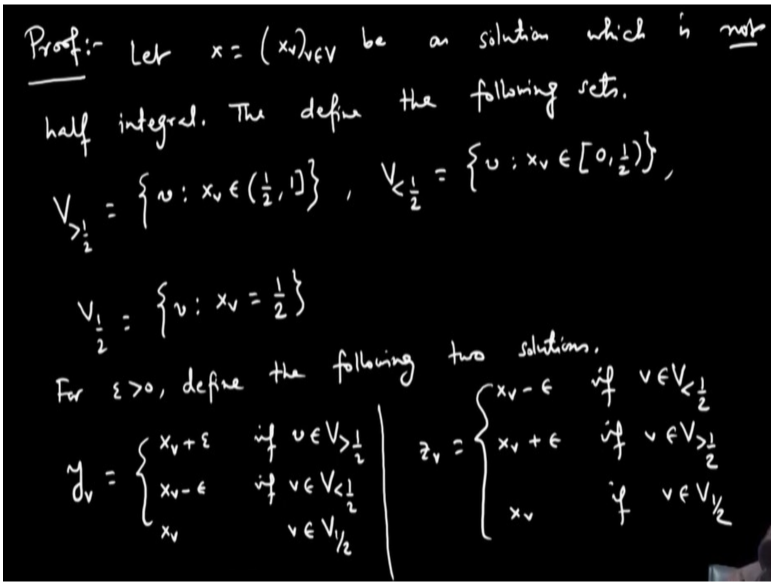

So 2 by contradiction so let $x=(x_v)v\in V$ and be an optimal solution which is not half integral. So then, define the following sets V greater than half is those vertices such that x of V belongs to half and one is more than half and V less than half v such that x of v belongs to 0 comma half. Of course V half a set of vertices v such that x of v is equal to half now using these sets we define another solution another 2 solution and show that this is a convex combination of those 2 solutions x, is a convex combination of those 2 solutions.

So for epsilon greater than 0 define the following to solutions so what is y_v is y another is $z_v=x_v+\epsilon$ if v belongs to V greater than half $x_v-\epsilon$. If v belongs to V less than half and this is x v if v belongs to V half this $y_v=z_v$. If v belongs to V less than half $x_v+\epsilon$ $x_v-\epsilon$ if v belongs to V less than half if v belongs to V greater than half and this is x_v if v belongs to V half.

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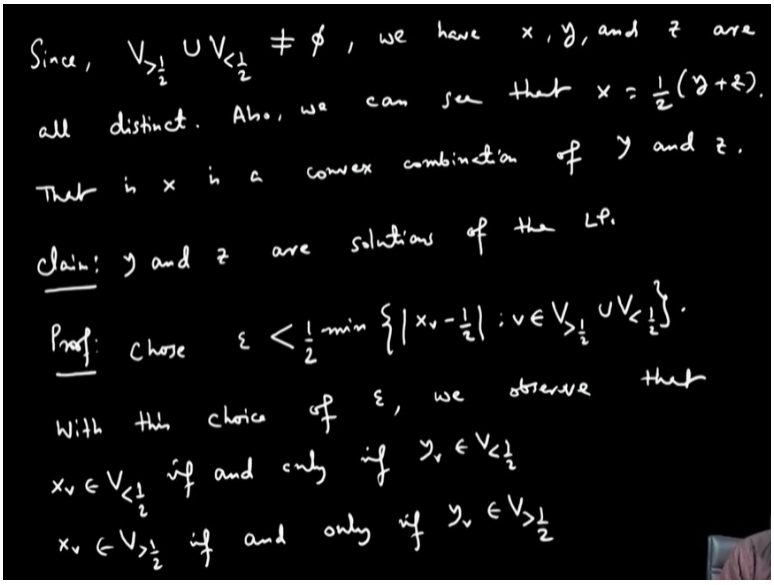

Now since v greater than half and there are some vertices where which x assigns more than half value and there is between a half and 1. So let us make this also because there are some values which on this part make it otherwise. So y_v add the value of ϵ, for those variables which are which are which belongs to v greater than half and reduces its value by epsilon for those variables whose value is in between 0 and half and z_v does exactly opposite.

Now because x_v x is not half integral there exists at least one vertex in v half v greater than half or v less than half that is that is since v greater than half union v less than half is not, equal to empty set x. Since this we have x y and z are all distinct also we can see that $x=\frac{1}{2}(y+z)$ that is x is convex combination of y and z. So define y = y_v in V and z = z_v in V so next we claim to finish the proof y and z are solutions of the LP it is proof.

First observe that we can choose ϵ small enough so that after this addition subtraction of, epsilon this vertices remain in corresponding sets. So choose epsilon less than what is the thing minimum of these sets x_v - half mod that x_v belongs to V greater than half union v less than half. So with this choice of epsilon so let me put a half also so with this choice of epsilon with this choice of epsilon we observe that x_v belongs to V less than half if and only if y_v, belongs to v less than half.

Same x_v belongs to V greater than half if and only if y_v belongs to v greater than half same with z_v.

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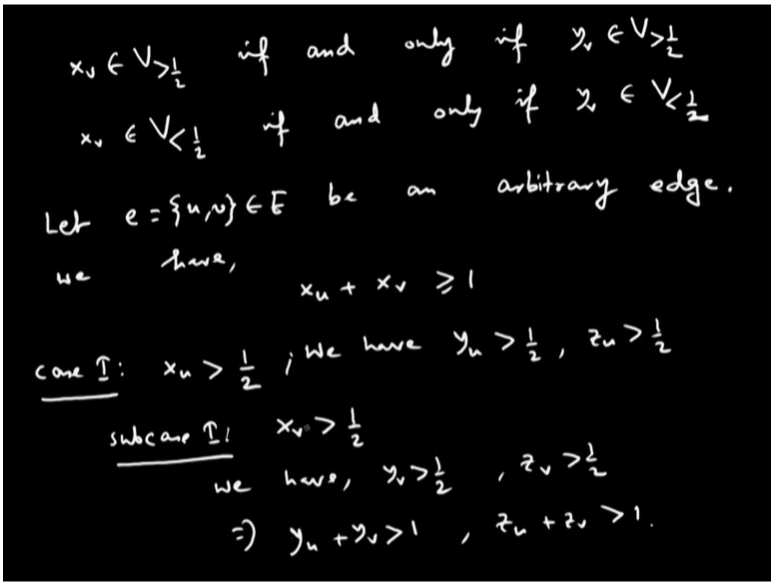

x_v belongs to v greater than half if and only if y_v belongs to v greater than half x belongs to v less than half if and only if y_v belongs to v less than half. So first of all then this new claim solutions y and z satisfy these constraints they are all positive. And also next we need to show that this each constraints are also satisfied with y and z. So let $e=\{u,v\}$ an arbitrary edge so we have x satisfy this constraint.

So $x_u+x_v \geq 1$. So what are the possible values of what are the possible cases. So case one you know x_u is greater than half if x_u is greater than half, then sub this x_v is also greater than half. Then we have in this case if x_u is greater than half then we have y_u is also greater than half and y_u, z_u is also greater than half. In this case if x_v is greater than half we have y_v is also greater than half and z_v is also greater than half.

Hence this implies $y_u+y_v \geq 1$ and $z_u+z_v \geq 1$ so for this, case if x_v is greater than half then it is then it is satisfied.

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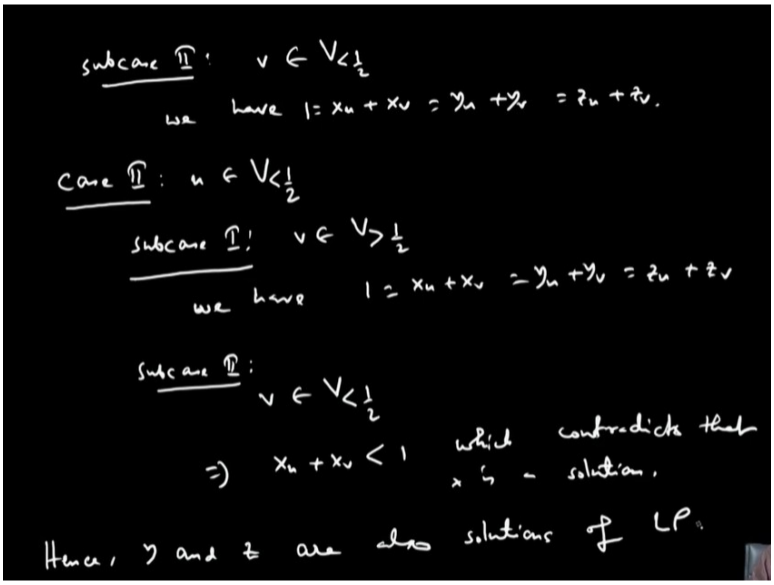

Let us do the next sub case x_v is less than half so first to fall next slide change if x_u belongs to v of v greater than half if x_u is 1 then y_u is 1 and z_u is also 1 nothing to prove. So this do this and also this is also x_v belongs to v greater than half. So in this case x_v belongs to v less than half in this case we have, v belongs to in this case we have $x_u + x_v = y_u + y_v = z_u + z_v$ and because $x_u + x_v$ is 1 in this case also the conditions are met this is v u.

So similarly the other surface can be done other case 2 u belongs to V less than half sub case one v belongs to V greater than half in this case also we have $1 = x_u + x_v$ which is equal to $y_u + y_v = z_u + z_v$. And hence the h constraint is satisfied sub case 2 you know sorry v belongs to V less than half but if both even we cannot belong to V less than half. If both values are less than half x_u and x_v is less than half this implies $x_u + x_v < 1$ which contradicts that x is a solution.

So in this case cannot happen in the other three cases we, have shown that this y and z also satisfies the constraint. So hence why and z are also solutions of LP but before we have already seen this is a convex combination so hence x is not an extreme point.

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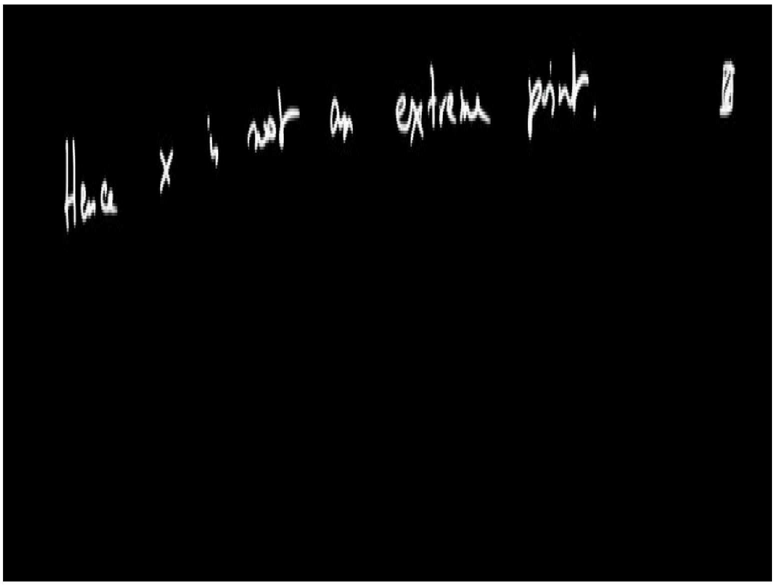

Hence x is not an extreme point which finishes the proof. So let us stop here using this exploit in this structure we will design a 2 factor approximation algorithm in the next class.