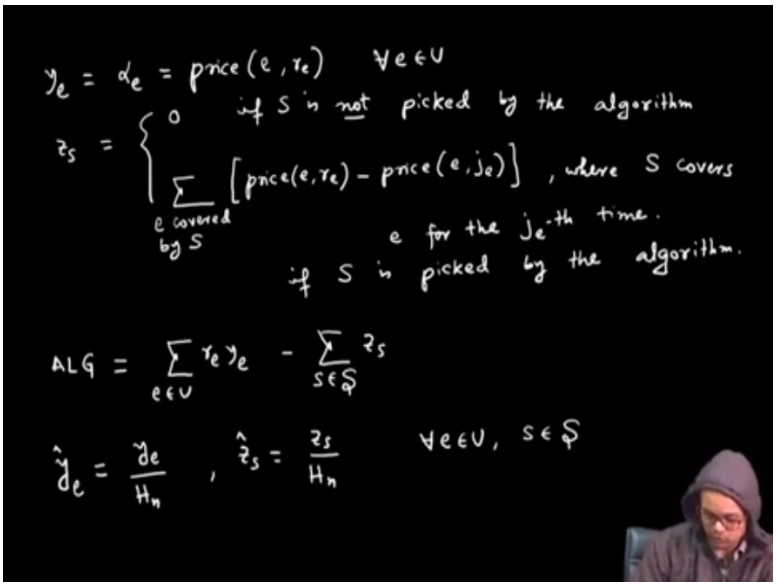
Selected Topics in Algorithms
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Module No # 11
Lecture No # 51
Dual Fitting

Thank you welcome so from the last couple of lectures we have been looking at how linear programming can be used for designing approximation algorithm designing and analyzing. And we have started looking at dual fitting. So in the last class we have shown we have seen that how dual fitting can be used to analyze the greedy algorithm for the basic set cover and the more generalized set multi cover, problem. So in that proof we left one part so, let us finish that part and then we will we will move on to next topic.

(Refer Slide Time: 01:06)



So dual fitting and there if you recall so the primal LP was minimize $\sum c(s)x_s$ subset of the collection s belongs to the collection subject to each element. For each element e is select at least r_e many sets that contain e is for all e in u that is 1. And then $-x_s$ is greater than equal to - 1 this is for all s in calories and x_s is greater than equal to 0. So this was primal LP and the dual was maximize $\sum r_e y_e - \sum z_s$ subject to no set is over-packed.

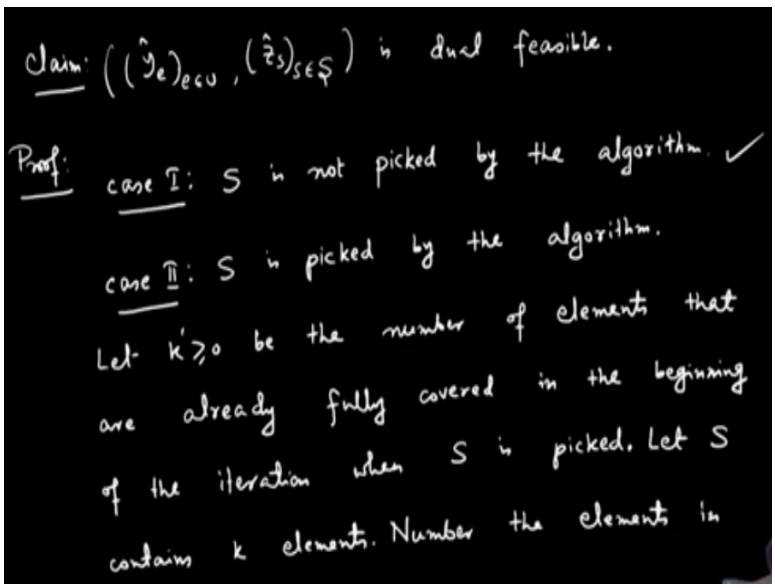
Means subject to for all $e \in S$, $y_e - z_s$ of is should be less than equal to $c(s)$ this is for all set is in script s and we, have y_e and z of is greater than equal to 0. For all element $e \in U$ for all $s \in S$, now to do dual fitting we looked at the setting of variable like α_e, y_e we set $y_e = \alpha_e$

which is like price of (e, r_e) this is for all $e \in U$ and z_s is 0. If s is not picked by the algorithm otherwise we sum over all elements e covered by s.

That means when s is picked that element e was, still alive price of (e, r_e) - price of (e, j_e) where j_e is the where s picked the s covers j_e when x is covered e for the j_e s type this is where s covers e for the jth time this is when is pigged by the algorithm. Then we have shown that l the cost of the solution picked by the algorithm is the value of dual objective function with this y_e and z is with this assignment.

So this is $\sum r_e y_e - \sum z_s$ then we define scaled this but this set of s this assignment for y and z is dual infeasible to make it feasible we scaled it with H_n. Define $\hat{y_e}$ to be $\frac{y_e}{H_n}$ and z is hat to be $\frac{z_s}{H_n}$ this is for all $e \in U$ and $s \in S$.

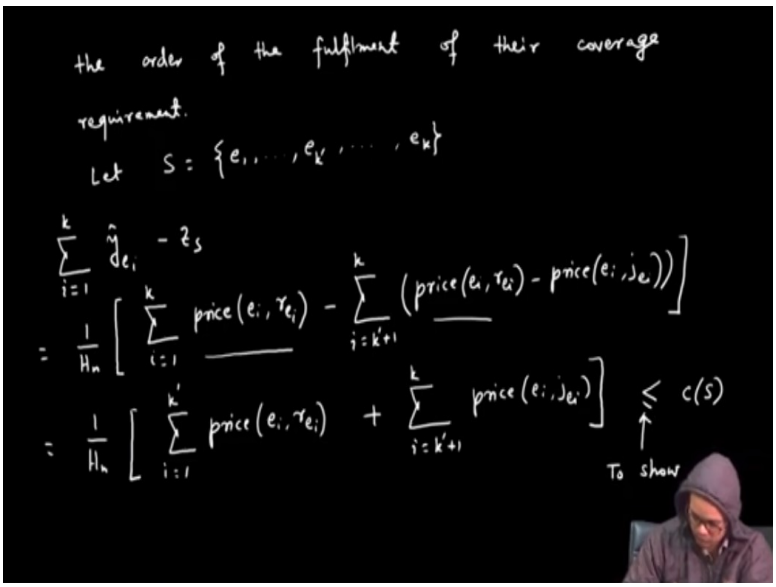
(Refer Slide Time: 08:25)



Now we need to show then we claimed that $\hat{y_e}$ $e \in U$ and $\hat{z_s}$ $s \in S$ is dual, feasible for that so what are the dual constraints. Let us look at we have a constraint for each set in the collection so we have 2 cases case 1 is not picked by the algorithm. So in this case we had shown that this constraints corresponding to s is satisfied for case 2 was remaining and this was given as homework.

Let us see the solution case 2 is S is picked by the algorithm so in this, case let us see so let us assume not assume let $k' \geq 0$ be the number of elements that are already fully covered in the beginning of the iteration when s is picked.

(Refer Slide Time: 12:30)

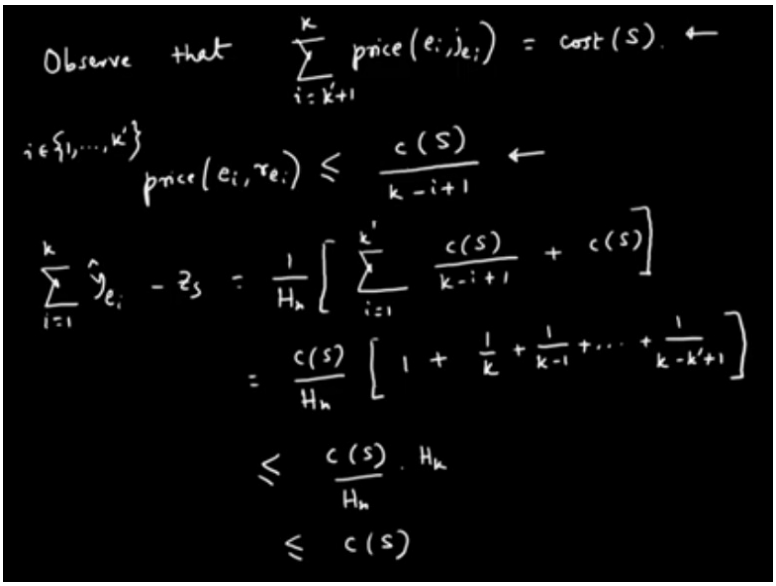


So let S contains k elements again like case 1 number the elements in the order of the fulfillment of their coverage requirement. Let S within $e_1,\ldots,e_{k'},\ldots,e_k$ so this, elements first $e_1,\ldots,e_k,e_{k'}$ they are already covered and S covers $e_{k'+1},\ldots,e_k$ so the cost of S will be equally shared by $e_{k'+1},\ldots,e_k$. So then let us see what is the summation what is the dual constraint for S?

The dual constant is $\sum \widehat{y}_{e_i} - z_s$, now what is this? This is $\frac{1}{H_n}\sum \widehat{y}_{e_i}$ is $\sum \widehat{y}_{e_i}$ is price of e_i in the r_{e_i} copy. So this is price by H_n this is price of (e_i, r_{e_i})-th copy - z_s, now z_s is c, over the elements covered by s this is from $i=k'+1,\ldots,k$ this is price of (e_i, r_{e_i}) - price of (e_i, j_{e_i}) where the set e covers the limit e_i for j_{e_i}th time.

Now let us see what we have this is $\frac{1}{H_n}$ the, $k'+1$ to k this price is gets canceled. And the first k prime element survive my, this plus $\sum price(e_i)$ is z_{e_i}. Now we need to show that this is less than equal to $c(s)$ this is to show.

(Refer Slide Time: 16:51)



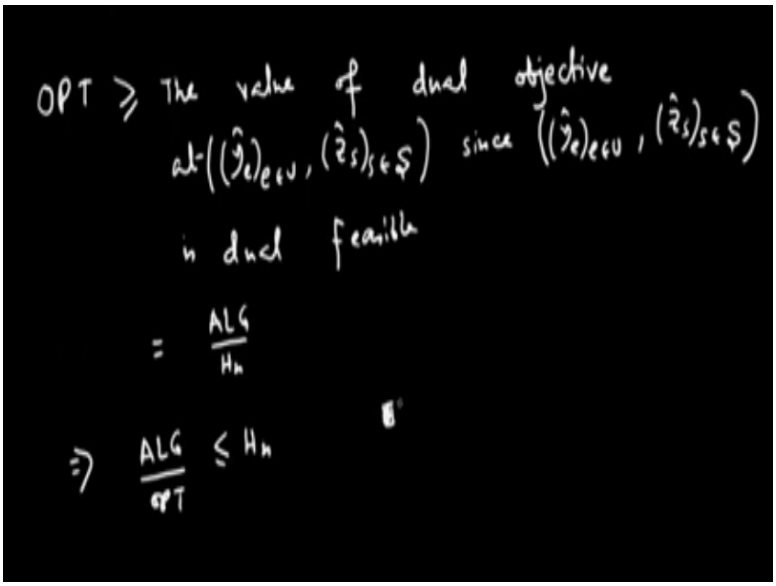
So let us show that first observe that S covers $e_{k'+1}, \ldots, e_k$ this elements. So the price of cost of S will be equally bond by this element so, this second term is at most $c(s)$ this is exactly $c(s)$ that is what we write $\sum price(e_i, j_{e_i}) = c(S)$. But let us see now we need to bound this sum what is price of (e_i, r_{e_i})? So you see that let us bound price of (e_i, r_{e_i}) this is for $i \in \{1, \ldots, k'\}$.

Now focus on any I when it is the it is covered for the last time this, set s was available to so suppose here is e_i somewhere here is e_i this set s was available to cover e_i and its cost will be born at least by this many elements that means k – i + 1. So but S was not picked some other set was picked so this must be less than equal to $\frac{c(s)}{k-i+1}$. So using these 2 we now put this inequality and this equality here and let us see what we get?

Then $\frac{1}{H_n}\sum_{i=1}^{k} \hat{y}_{e_i} - z_s$ this is $\sum_{i=1}^{k'} \frac{c(S)}{k-i+1}$. So this is what is this sum this is let us take $c(S)$ common outside. Now this sum is at most H_k and some more terms.

So this is less than equal to $\frac{c(S)}{H_n} H_k$ but H_k increases with k so this is less than equal to $\frac{H_k}{H_n} \leq 1$ this is less than equal to $c(S)$ which concludes the proof.

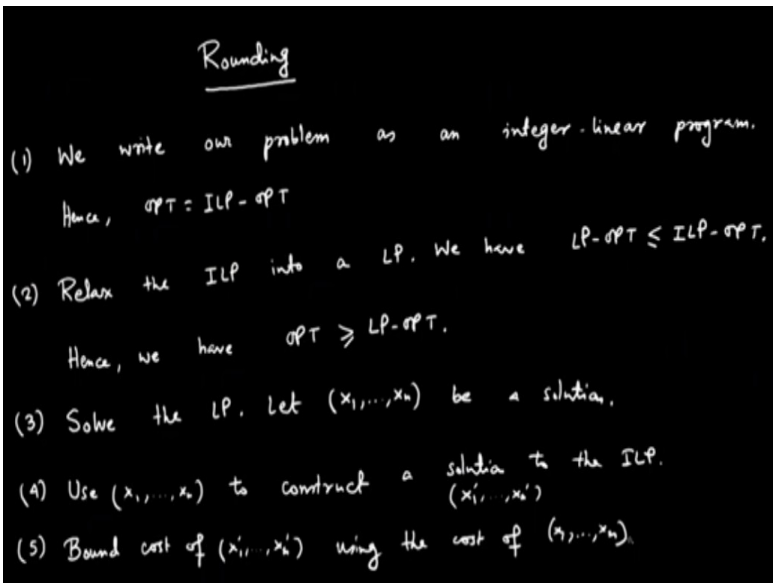
(Refer Slide Time: 21:51)



So using this we have shown hence and then the last part we have shown that how using this, the last part is because this is a dual feasible solution. So opt is greater than equal to the value of dual objective at $\left((\hat{y}_e)_{e\in U}, (\hat{z}_s)_{s\in S}\right)$ the value of the dual objective at this since this solution $\left((\hat{y}_e)_{e\in U}, (\hat{z}_s)_{s\in S}\right)$ is dual feasible. But then this is $\frac{l}{H_n}$ so hence $\frac{ALG}{OPT}$ is less than equal to h n k this proves the approximation factor of the algorithm.

So this shows the first technique of analyzing a combinatorial algorithm or a different algorithm for a problem using linear programming duality the next approach is rounding.

(Refer Slide Time: 23:58)



So which; is more conventional approach for using linear programming in designing approximation algorithm rounding. So what; is the idea that we write step 1 so step 1 we write our problem as an integer linear program. Hence we have opt equal to ILP opt next to a relax

the ILP into LP we have LP of if suppose it is a minimization problem write it in a standard form in a minimization format we have LP opt then is less than equal to ILP opt why?

Because we are minimizing the same objective but the search space is a superset in LP compared to the ILP. So this hence we have opt is greater, than equal to LP opt so this gives the lower bound that we will be comparing with and third this approach requires solving linear program. Solve the LP let $(x_1,\ldots,x_n)$ be a solution because $(x_1,\ldots,x_n)$ is a solution to LP it need not be a valid solution for ILP.

So use LP use this solution $(x_1,\ldots,x_n)$ to construct solution to the ILP hence that will be a valid solution for the problem at hand. Finally solution let us call it $(x_1^{'},\ldots,x_n^{'})$ bound cost of $(x_1^{'},\ldots,x_n^{'})$ using the cost of $(x_1,\ldots,x_n)$. So this approach we will see in the next class with some other some problems thanks.