Selected Topics in Algorithm
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Lecture – 05
Flow Decomposition

Welcome so, in the last class we have finished discussing Edmond-Karp algorithm and we have also proved the max flow mean cut theorem.

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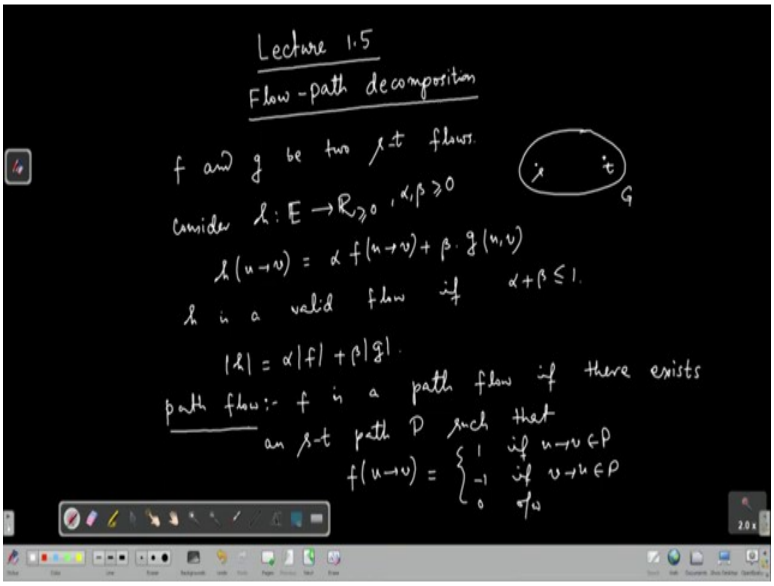

So, in this lecture we will study another algorithm for max flow another natural algorithm and some more properties, this is called flow path decomposition. So, suppose I have a flow graph G with source vertex s and destination vertex t and suppose I have two flows. So, f and g be two s to t flows. Then consider the function consider h define from edge to real number $h(u \to v) = \alpha f(u \to v) + \beta g(u \to v)$ where α, β , are non-negative numbers.

And one can it is easy to verify that h is a valid flow if $\alpha + \beta \leq 1$ and moreover, the value of this flow is $\alpha f + \beta g$. Now, it turns out that every flow can be written as a combination of some elementary flows which are called path flows. So, if path flow, what is path flow? Path flow is f is a path flow if there exists an s to t path P, such that all the edges in this path carries positive flow f of u to v. This is one if u to v belongs to the path, –1 if v to u belongs to the path and 0 otherwise.

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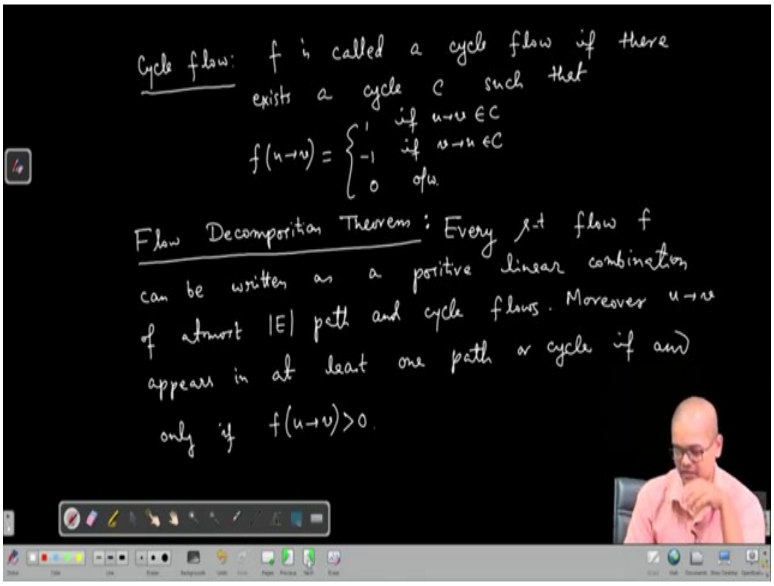

And along the path flow along similar lines, we can define cycle flow. Then f is called cycle flow if there exists a cycle c such that $f(u \rightarrow v)$ is 1 if u to v belongs to the cycle –1 if v to u belongs to the cycle and 0 otherwise. Now, the fundamental theorem of flow decomposition is that any flow can be written any s t flow can be written as a linear combination of small number of path flows and cycle flows these are elementary flows.

So, flow decomposition theorem every s t flow f can be written as a positive linear combination of at most size of E many path and cycle flows. Moreover, an edge u to v appears in at least one path or cycle if and only if it carries a positive flow.

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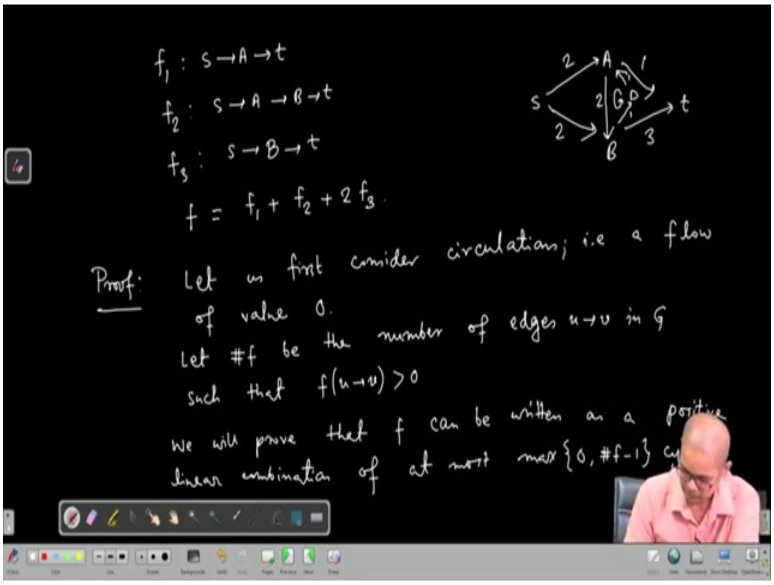

So, before proving it, let us see an example, so, suppose S to A to B t and suppose the flow values are say 2, 1, 1, 1. Suppose these are the flow values. Then let us consider three path flows S to A to t this is, let us call it f_1 , f_2 is S to A to B to t let us make it slightly more non trivial and f_3 is S to B to t. Then we can write f as $f_1+f_2+2f_3$. So now, let us prove the theorem.

The proof is in some sense constructive only let will find the flow path on an S to t path where each edge carries a flow and we will try to push as much flow along that path as possible and that will free up that will make the flow in an edge to in the residual flow in an edge 0. And if we iterate over E many times then we should be able to free up all the edges. But what will happen?

If there is no S t path or if the current flow value is 0. So, there could be cycles. For example, what if there is an edge here like D? And it is possible that some unit of flow is circulating here. So, this will happen if you know the flow values of A B is 2, B D is 1 and A D is 1. So, to decompose this, we also need this cycle A B D. So, these are called circulations and let us consider those circulations first.

So, circulation is a flow value is 0. So, let us first consider circulations that is a flow of value 0. So, it simply respects flow conservation, property and of course, the capacity constraint. So, let us define let hash f be the number of edges u to v in G such that f of u to v is positive number of edges which carries some positive amount of flow. So, what we will prove? So, we will prove that f can be written as a positive linear combination of at most max of 0 and hash f −1 cycles.

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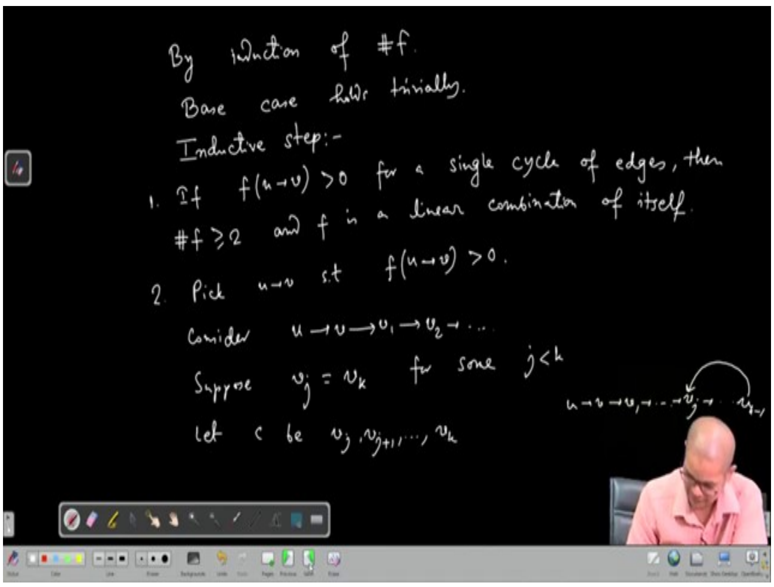

So, we will prove it by induction on #f if #f is 0 then I am supposed to write f as 0 many as a positive linear combination of 0 many cycles which is trivial is true. So, base case holds trivial. So now, let us do the inductive step so, by induction I need to prove for hash f, if hash f many. So, I need to write it in if hash f for 0 to #f –1 suppose this statement is true and I will prove the inductive step.

So, couple of cases, if f of u to v this is 0 for a single cycle of edges. It means the edges which carry positive amount of flow. If that forms a single cycle then it is very easy then to form a cycle because there is no antiparallel edges hash f is greater than equal to 2 and f is a linear combination of itself. Second condition so the set of edges which carry positive amount of flow, they do not create a cycle, they do not make a cycle.

So, let us pick any edge u to v such that f of u to v is greater than 0. So now, let us consider a work u to v but because of the flow conservation property at v there must be an outgoing flow edge. So, let us call it say v_1 and again because of the flow conservation property of v_1, it is go to v_2 and so on. And because the number of vertices is finite, it must cycle so, all these edges carry positive amount of flow and suppose $v_j = v_k$ for some j less than k.

That means here is u, v, v_1 and here is v_j goes on and this is $v_{k-1} v_j$. So, I have got a cycle, so, I will use this cycle to reduce the flow value and the number of edges carrying positive amount of flow and use induction, so, let us do that.

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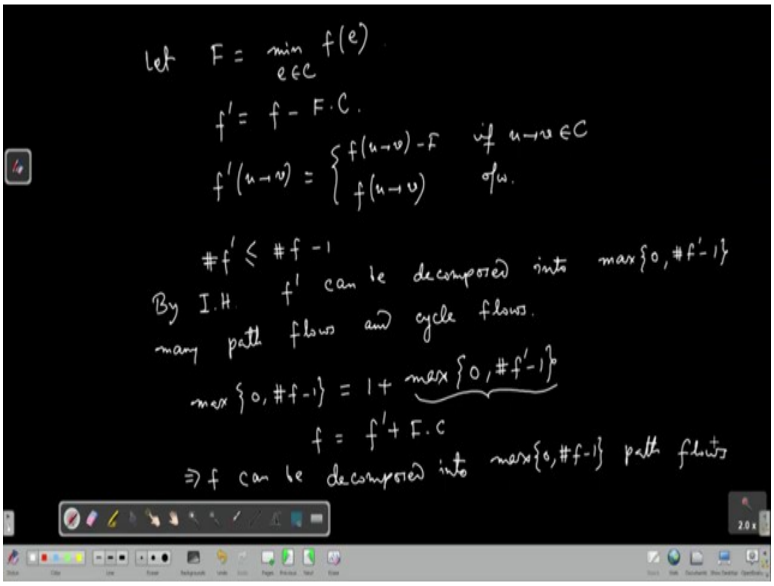

So, let define let capital F be minimum over edge in c. So, let us call c be the cycle let c be v_j , v_{j+1} to v_k this cycle. So, what is the minimum flow value of a flow that any edge is carrying? So, minimum of f of u what is it? It f_e so, this much flow value we can subtract, so, we can write f' equal to f – capital F times this cycle C this cycle flow. And so, how it is defined? f' of u to v is f of u to v– f if u to v belongs to this cycle.

Otherwise, it is same as $f(u,v)$. Now, observe that number of edges carrying positive amount of flow is in f' is at most number of edges carrying positive amount of flow in f – 1 because any edge in c which carries capital F amount of flow that particular edge does not carry any flow in f' . So now, by induction hypothesis, f prime can be decomposed into maximum of 0 and # f' –1 many path flows and cyclic flows.

But max of 0 and #f –1 is 1 + max of 0 # $f'-1$. And f is nothing but f' + capital F times c so and f' can be decomposed into max of 0 and # $f'-1$ many path flows and cycle flows. Then f can be decomposed into a max of 0 and # f –1 path flows and cycle flows.

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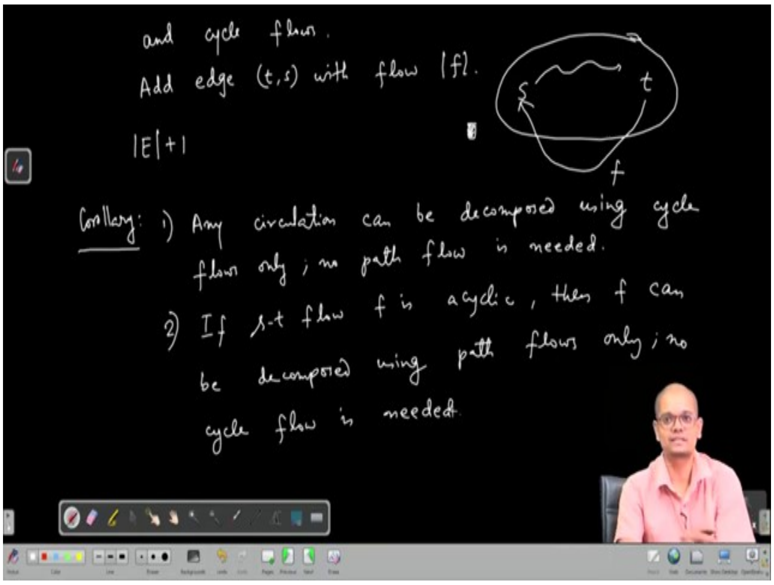

So, there are two immediate corollaries which follow from the proof any circulation, no just a minute, so, we have proved this for circulation only. We need to prove for paths so, here also, we can observe that these are all only c is a cycle. So, if even we do not need paths also, this many cycle flows are enough. If the original given flow is a circulation then this many cycle flows are enough.

So, this proves the result for circulations high volt for any arbitrary flow. So, suppose I have a flow and s to t. What I do is that? I have a certain flow in a flow graph. I add and new edge from t to s and make it slow f so, add edge t s and with flow value of f of t. So now, you see that the flow conservation property is satisfied even at s and t and the flow value is 0 and. now we have a circulation.

So, this new flow this circulation can be decomposed into at most E + 1 edges. Because then we have added a new edge. So, currently the number of edges number of the cardinality of E + 1 that is the maximum number of edges which can carry some non negative flow. And with this many circulations with this many cycle flows, this circulation can be decomposed and from all those cycle flows which contain that t to s edge.

If we delete t to s flow then what we get is flow paths? And that is how we get the result for paths also for arbitrary flow. So, we get two important corollaries first, is any circulation can be decomposed using cycle flows only. In particular no path flow is needed that is first point. Second point is if this s t flow f is acyclic then all when we add this reverse edge t to s all these circulations all these cycles will involve this edge s to t.

And hence at the end, when you remove it, this flow f is decomposed into path flows only. So, if the s t flow f is acyclic then f can be decomposed using path flows, only no cycle flow is needed. So, in the next lecture we will see how using this flow decomposition? We can have another algorithm of maximum flow and with it is running, time will be incomparable with the running time of Edmonds-Karp algorithm. Thank you.