Selected Topics in Algorithm

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Lecture - 47
Introduction to Linear Program

Welcome, so in the last couple of lectures we have been looking at approximation algorithms. So, let us briefly summarize what are the algorithms we have seen so far, what are the approximation algorithms.

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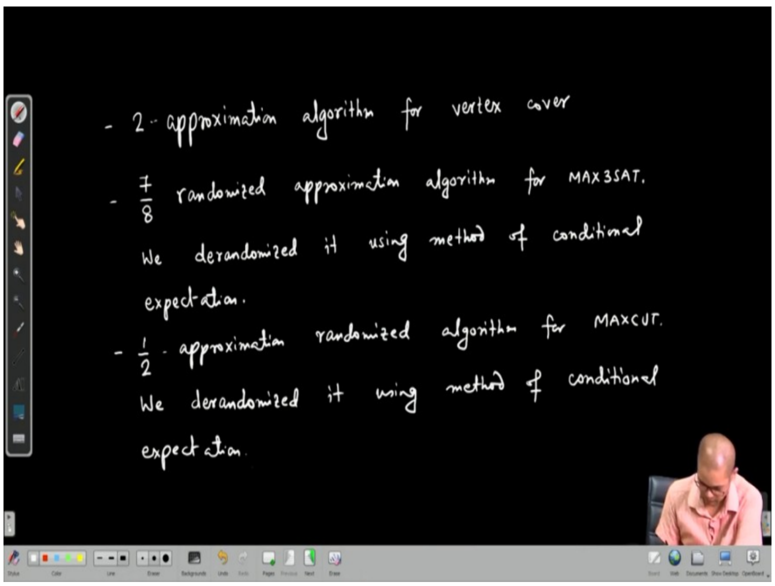

So, we have seen a two factor two approximation algorithm for vertex cover, we have seen a $\frac{7}{8}$ randomized approximation algorithm for MAX3SAT. We also de-randomized it using method of conditional expectation then we have seen a half factor approximation algorithm, half approximation algorithm, approximation randomized algorithm for MAXCUT. We de-randomized it using method of conditional expectation.

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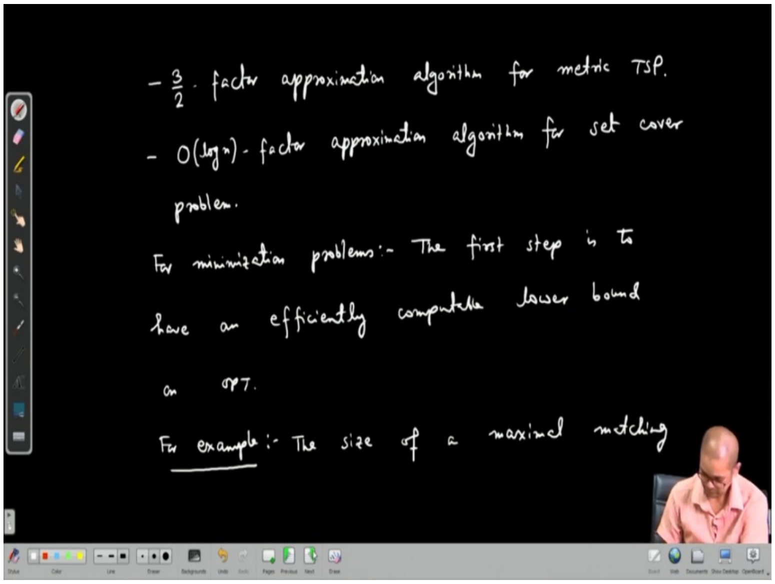

Then we have seen $\frac{3}{2}$ factor approximation algorithm for metric TSP which is a version of travelling salesman problem where triangle inequality is satisfied. Then we have seen $O(\log n)$ factor approximation algorithm for set cover problem. So, all these algorithms or the design does not follow a systematic approach, it you know each algorithm needs a new idea and but one sort of high-level idea is for minimization problems.

The first step for an approximation algorithm so, to say is the first step is to have a computable efficiently computable lower bound on OPT. For example, you know the lower bound that we have used for vertex cover is matching. So, the size of maximal matching well is lower bound on vertex covered.

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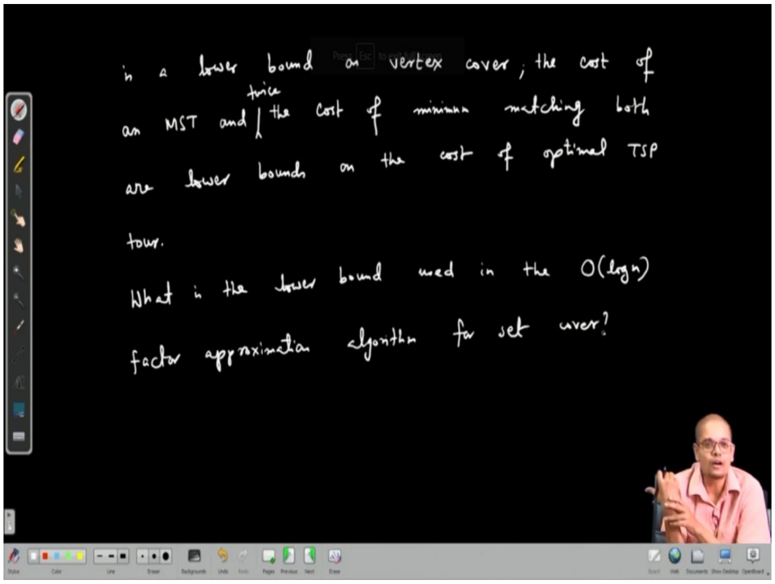

Then for a travelling salesman tour we have used two lower bounds, the cost of an MST minimum spanning tree and the cost of minimum matching and you know two times twice the cost of minimum matching both are lower bounds on the cost of optimal tool, optimal TSP tour. So, but what is the lower bound for set cover using which we have derived at $O(\log n)$ time algorithm.

What is the lower bound used in the $O(\log n)$ factor approximation algorithm for set cover? So, we will see that you know that a lower bound for this problem comes actually from some what is called a linear program and not only that, that linear programming framework allows us to systematically come up with lower bounds on any optimization problem. So, our next topic is a linear programming-based approximation algorithm.

And this sort of algorithms will be much more systematic than these algorithms this approximation algorithms that we have seen so far. These algorithms are sort of look like ad hoc and each algorithm need sort of a new idea.

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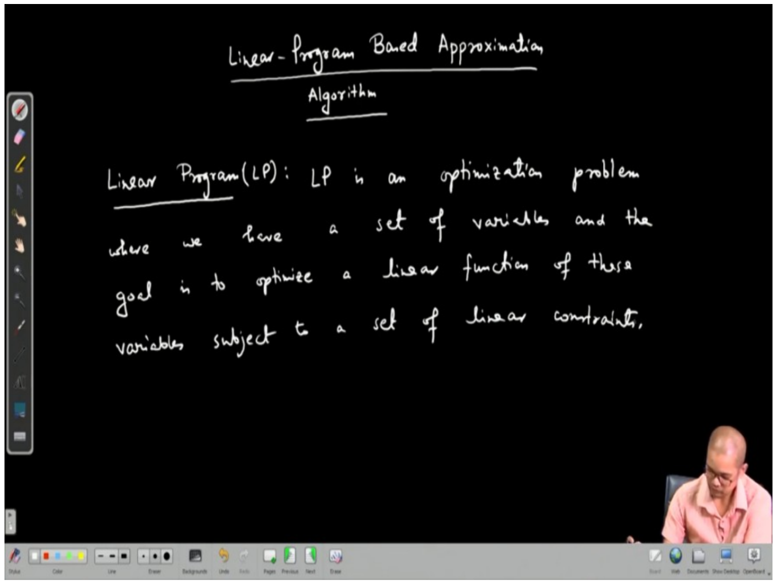

So, we will the next topic that we look at is linear program-based approximation algorithm. So, to look at this, so let us briefly discuss what is a linear program. So, linear program; now linear program or LP, in short LP is an optimization problem where we have set of variables and the goal is to optimize linear function of these variables subject to a set of linear constraints.

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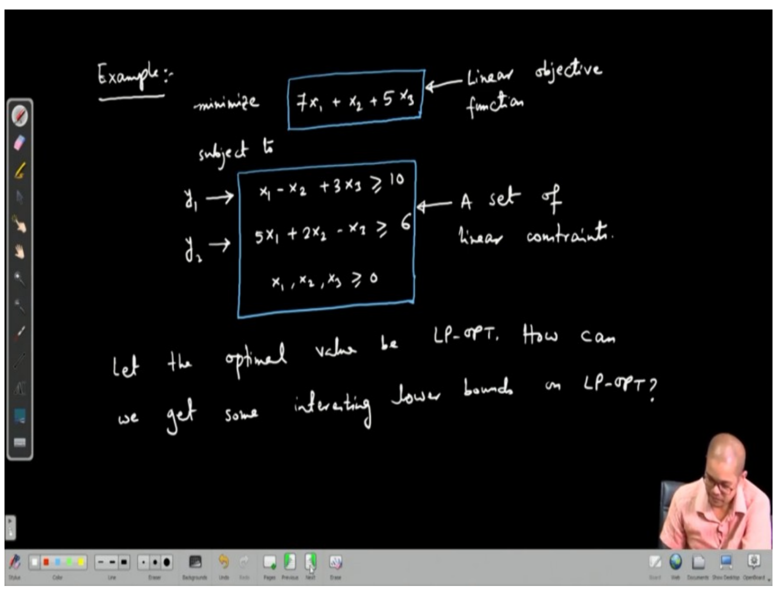

So, for example minimize say $7x_1+x_2+5x_3$ subject to $x_1-x_2+3x_3\geq 10$, $5x_1+2x_2-x_3\geq 6$ and x_1,x_2,x_3 this is greater than equal to 0. So, this is the linear optimization for objective function which we want to maximize in this case and these are the linear constraints. This is linear objective function and this is a set of linear constraints. So, given such a linear program one is interested to find interesting lower bounds.

So, lower bounds on the optimization or of the optimum value, so let the optimal value LP-OPT. How can we get some interesting lower bound on LP-OPT? For that you know we can multiply these constraints with some number and then try to maximizes. For example, suppose let me multiply this first inequality with y_1 and second inequality with y_2. Then what do I get?

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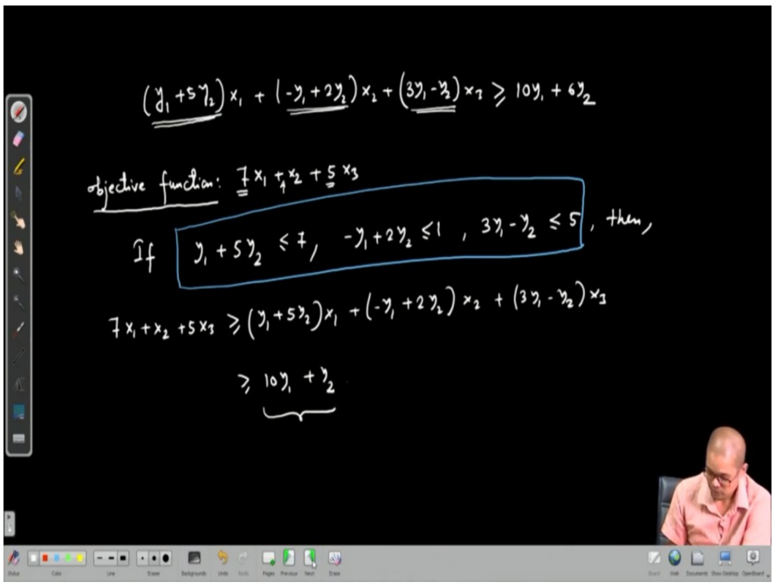

I get $(y_1+5y_2)x_1+(-y_1+2y_2)x_2+(3y_1-y_2)x_3$, this is greater than equal to $10y_1+6y_2$ and what was my objective function, my objective function was $7x_1+x_2+5x_3$, so this was my objective function. So, now if it happens that y_1+5y_2 is less than equal to $7-y_1+2y_2$ is less than equal to 1 that means the coefficient of x_1 here is less than equal to coefficient of x_1 here and coefficient of x_2 here is less than equal to coefficient of x_2 which is 1.

And coefficient of x_3 is less than equal to coefficient of x_3 here. If this happens then we can write $7x_1+x_2+5x_3$ is greater than equal to $(y_1+5y_2)x_1+(-y_1+2y_2)x_2+(3y_1-y_2)x_3$ but this is greater than equal to $10y_1+y_2$. So, for each combination of y_1, y_2 which satisfies these constraints these three constraints gives us a lower bound the corresponding y_1, $10y_1+6y_2$ is a lower bound on OPT.

The OPT is must be at least $10y_1 + y_2$. Now which will be the strongest lower bound the corresponding y_1 and y_2 which will maximize this.

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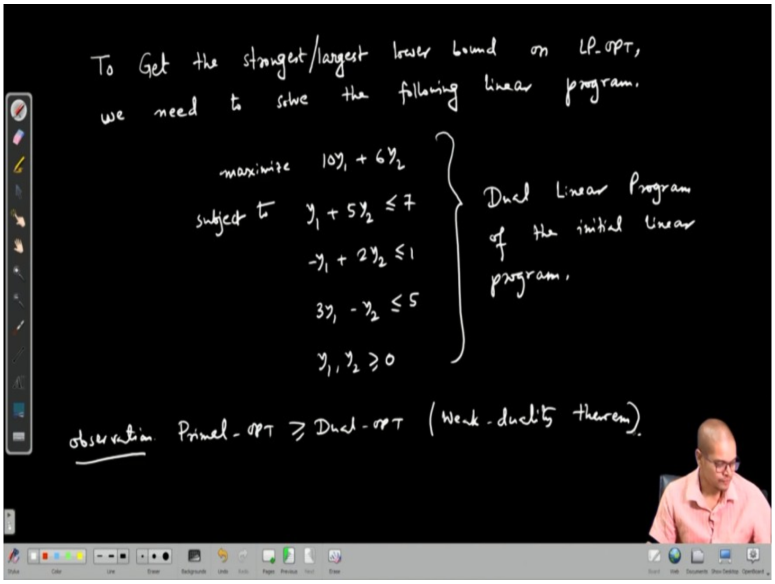

So, to get the strongest which basically means largest lower bound on LP-OPT we need to solve the following linear program. We want to get a strongest lower bound that means the largest lower bound that is y_1, y_2, maximize $10y_1 + y_2$, so maximize $10y_1 + y_2$ subject to the constraints. Here $10y_1 + 6y_2$ these are lower bound only if all the three conditions in this box is satisfied that means $y_1 + 5y_2$ is less than equal to $7 - y_1 + 2y_2 \le 1$ and $3y_1 - y_2 \le 5$ ..

And because we have multiplied these equations with y_1 and y_2 without changing the inequalities that means y_1 and y_2 must be non-negative numbers. So, this program is called the dual linear program of the initial linear program. The initial linear program is sometimes called the primal linear program. From the discussion, it is obvious that primal OPT is greater than equal to dual out is observation.

And this is what is called the weak duality theorem which you will formally state under certain conditions. So, this holds if and only if you know these linear programs both these are not unbounded and it has an optimal solution. So, under normal circumferences this is the condition which is basically called weak duality theorem.

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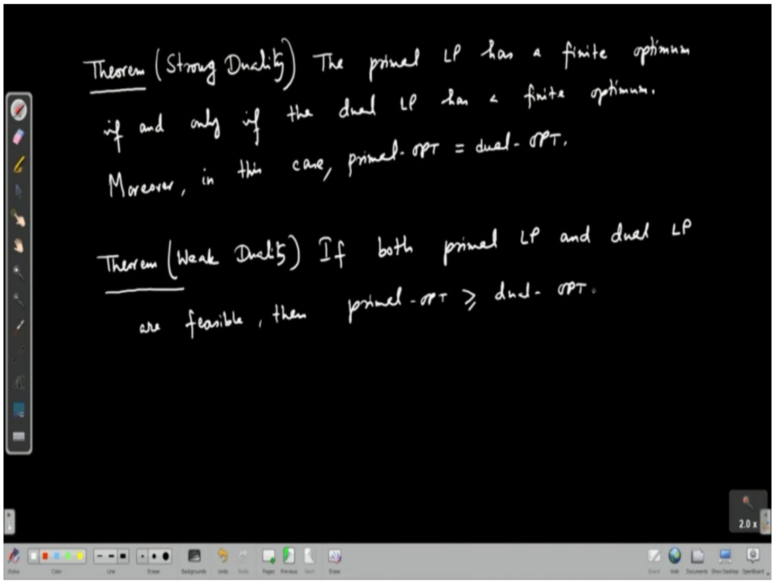

But not only that we have stronger result that the volatility theorem says that you know these two coincide the optimal solution of dual is the same as the optimal solution of primal. So, here is let me state theorem is called strong duality theorem. The primal program the primal LP has a finite optimum if and only if the dual LP has a finite optimum. In this case primal OPT = dual OPT. So, this is the strong duality theorem.

And we also have what is called a weak duality theorem which simply follows from the construction of dual linear program weak duality. If both primal LP and dual LP are feasible then primal OPT is greater than equal to dual OPT and also. So, in the next class we will formally define duality or and what is the standard form of a linear program and then we will also show that you know dual of dual is a primal. So let us stop here today, thank you.