Selected Topics in Algorithm

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Lecture - 36
NP – Completeness of 3 - Colouring

Welcome. So, we have been doing seeing NP completeness reductions and let us see what are the reductions we have seen till now.

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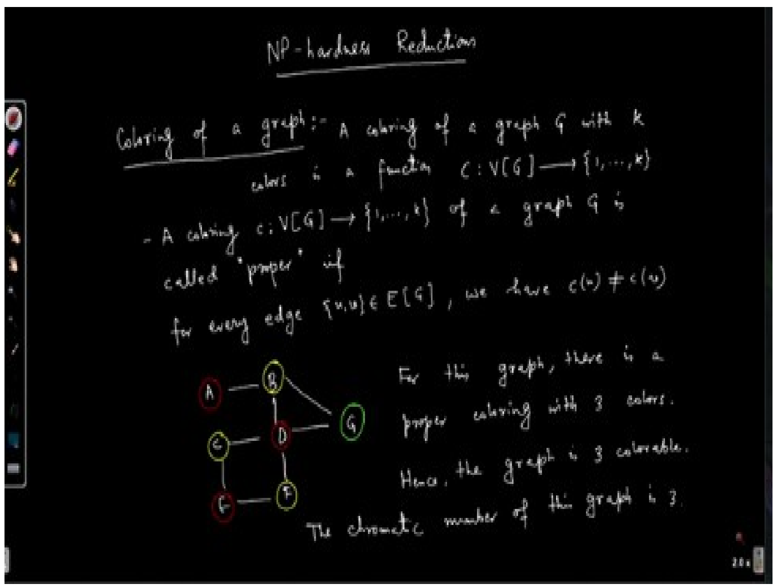

And we will continue some more reductions. So, NP hardness reductions. So, in today's class we will see NP completeness of 3 colouring problem. So, what is colouring of a graph? So, typically by colouring of a graph we mean colouring the vertices of a graph. So, a colouring of a graph G with say k colours is a function say C from the set of vertices to one of these k colours. A colouring is called proper colouring C from $V[G]$ to $\{1, \ldots, k\}$ of a graph G is called proper.

If each edge sees two colours for each edge its endpoints are coloured with two different colours. So, proper if for every edge $\{u, v\}$ in the E set of G for every edge u, v we have colour of u is not colour of v. So, let us take an example. So, A, B, C, D, E, F, G so suppose this is the colour this is the graph and let us colour the vertices with pure colour as possible. Let us try to properly colour the vertices of G.

Of course, if we colour each vertex with a separate colour that is a proper colouring but it invokes in many colours so in the colouring problem the goal is to colour the vertices with as few of colour as possible. So, let us try to colour. Let us colour D with red and let us see if we can reuse this colour B colour red, this A also can be coloured red and E also can be coloured red. Now let us try to colour now B can be coloured yellow, C could be coloured yellow and F could be coloured yellow.

And G you look G could not be coloured red or yellow so G let us colour it G. So, we have properly coloured with 3 colour. So, for this graph there is a proper colouring with 3 colours. Hence the graph is 3 colourable. Can this graph be coloured with 2 colours? No, because there exists a triangle BDG is a triangle colouring these three vertices must be given three different colours hence the graph is not too colourable.

Three is the minimum number of colours needed to properly colour this graph and that this minimum number is called chromatic number, the chromatic number of this graph is 3.

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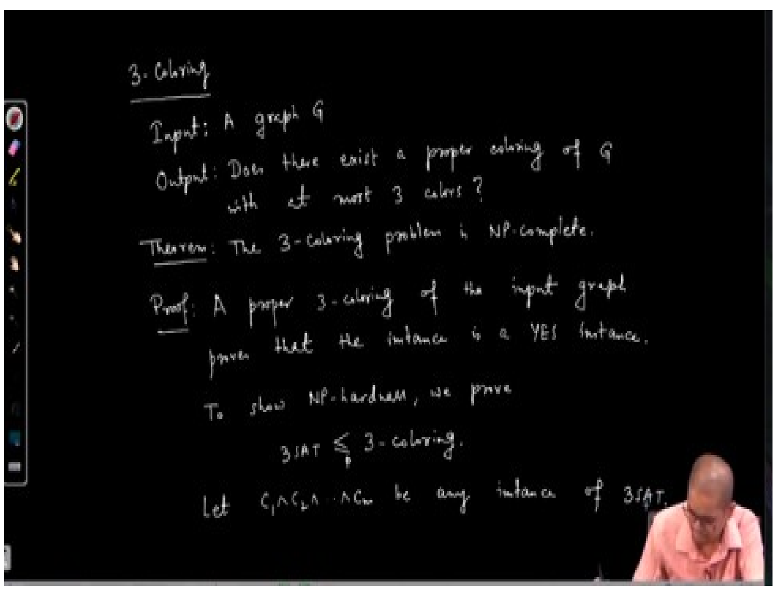

So, for any integer k we can define the k colourability problem and the question the given a graph G is does there exist a proper colouring with k colours. In particular we will see 3 colourability problem or 3 colouring problem. So, what is the 3-colouring problem? The input is a graph G, output it is a decision problem. So, it should be posed as a yes, no question. Does there exist; a proper colouring of G with 3 colours with at most 3 colours.

So, we will show that this problem is NP complete. So, theorem the 3 colouring problem is NP complete k, it has two parts one is membership in NP and the other is NP hardness. Now membership is again easy to show that some problem instances and each instance is a valid 3 colouring of the graph is a certificate. So, a valid 3 colouring or proper you know valid and proper they used interchangeably but let us stick to one name proper.

A proper 3 colouring of the input graph proves that the instance is yes instance. Now to show NP hardness we will reduce from 3SAT. To show in P hardness we prove that 3SAT polynomial time may need to reduce to 3 colouring. So, let $C_1 \wedge C_2 \wedge ... \wedge C_m$ be any instance of 3SAT.

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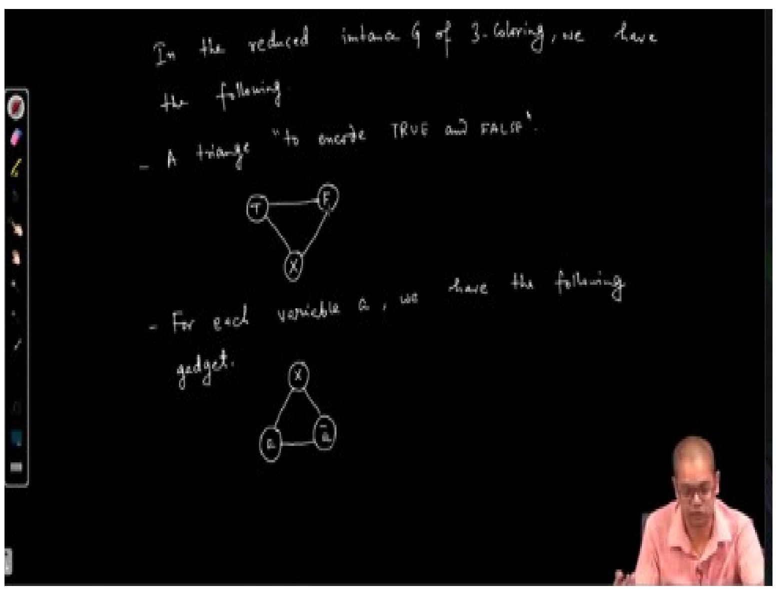

So, in that the reduced instance G of 3 colouring we have the following. So, we have a triangle informally saying to encode true and false. So, what is the triangle? So, let us T, F were vertex level T, I have a vertex level F and I have a vertex level X which means arbitrary. Next for each variable X we have the following gadget. So, this X from this X vertex these variables I have a vertex level X and I have a vertex level X bar.

So, let us call this a because X is already used a, a bar. Now you see that the colour of E should be different of the from the colour of X and the colour of a bar should also be different from the colour of X, not only that a and $\bar{a}$ their colour also should be different from each other. So, this is how we will encode true and false. It is like the colour of a if it is C as the colour of T that means variable a is set to true.

And if variable a is set to true if that that means if colour of a is same as colour of T and we have only 3 colour that means colour of a bar should definitely be colour of F that means a bar is set to false. So, that is the intuition we will formally prove it.

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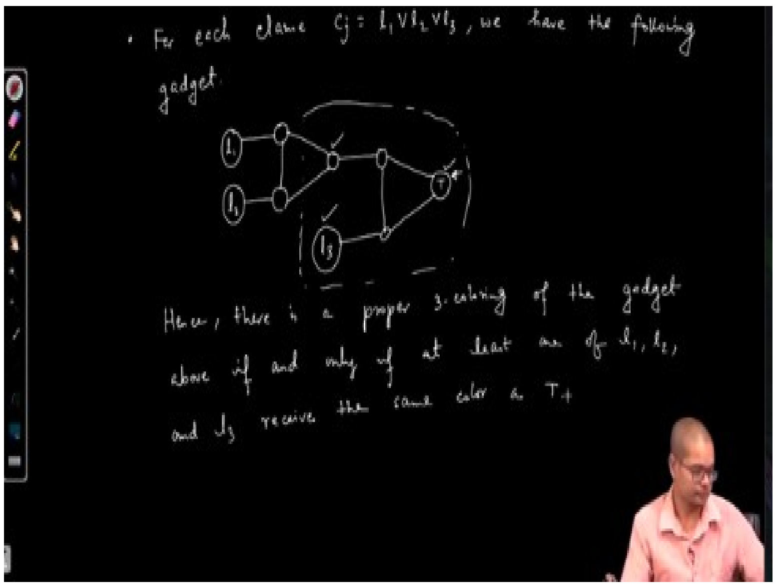

So, for each vertex for each variable we have get it now for each clause. For each clause C_j equal to it is an or of three literals l_1, l_2, l_3. We have the following gadget. So, what is it? So, from l_1 and l_2 these are gadget it is slightly non-trivial, let us see. So, this vertex the colour of this vertex should will be the majority of l_1 and l_2. If both l_1 and l_2 are coloured say true then this vertex must also be colour true.

Let us verify it, both l_1, l_2 are coloured true that means let us call these vertices A B C then A and B their colours must be false and the colour of X. The colour of X let us call it X so A should be coloured say A and B together should be coloured as false and X that means C must be coloured as true. So, for other case also it is easy to verify that you know the colour of C must be the colour of the majority of l_1 and l_2.

That means if above l_1 and l_2 are coloured true then C must be colour true if both l_1 and l_2 are coloured false C must be coloured false. And if one of l_1 is coloured through l_2 is coloured false then let us see that then suppose l_1 is true and l_2 is false then C can be coloured both true or false both is possible. So, for example you know if we want to colour C if then I would colour B T and A X.

On the other hand, if I want to colour C as true then I need to colour B as X and A as A X also. So, C must be the majority of l_1 and l_2 so let us erase this. We have not used l 3 yet which we will use now. So, l 3 is here is l 3 and these are connected to true vertex. So, there is a true vertex here so all these clause gadgets are connected to this true vertex. So, that means that there will be a proper colouring for this gadget with 3 colours even only if at least one of them l_1, l_2 or l 3 is true.

So, hence because again this part of the gadget is a majority of this and this and this must be true. So, that means either l 3 is true or this should be true so that means majority or at least one of l_1, l_2, l 3 must be true. Otherwise, because this vertex is already neither true vertex otherwise, they should not will not be any proper 3 colouring. Hence there is a proper 3 colouring of the gadget above if and only if at least one of l_1, l_2 and l 3 receives the same colour as T. So, let us see an example that way it will be clear.

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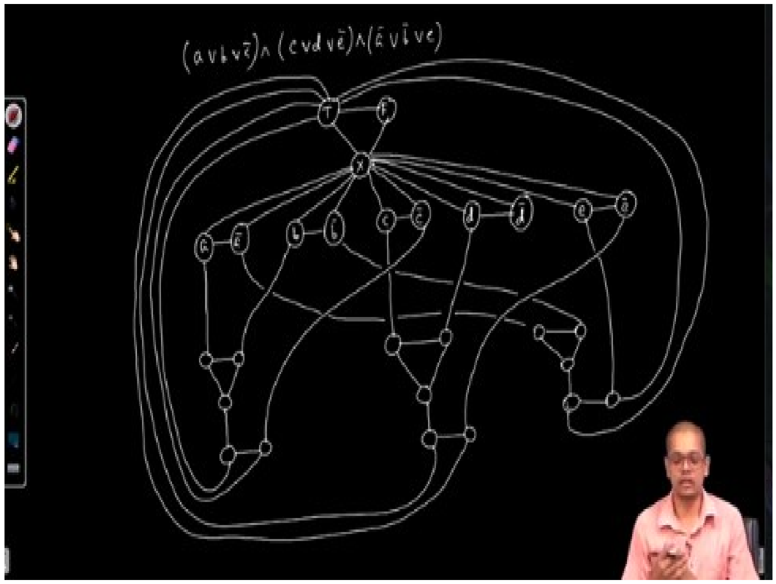

So, let us suppose this is a 3SAT clause say maybe a or b or c bar and suppose c or d or e bar and suppose a bar or b bar or e. Suppose this is my 3SAT formula so I have a gadget for true false and X it is a triangle. Now for each variable the variables are a, b, c, d, e I have these things I have a a bar, b b bar, c c bar d d bar and e e bar. And now I have these clauses so, for a, b what I have a e or b or c bar.

So, from a or b or c bar this is from c bar and this should be joined to T and these also join to T. So, this is the first clause the second clause is c d. So, let us draw here c d and then e bar and then these two are joined to T. And I have a bar, b bar e so a bar b bar these joined together is joined with e and then both of them are joined to T. So, this is how the converted graph the reduced graph looks like.

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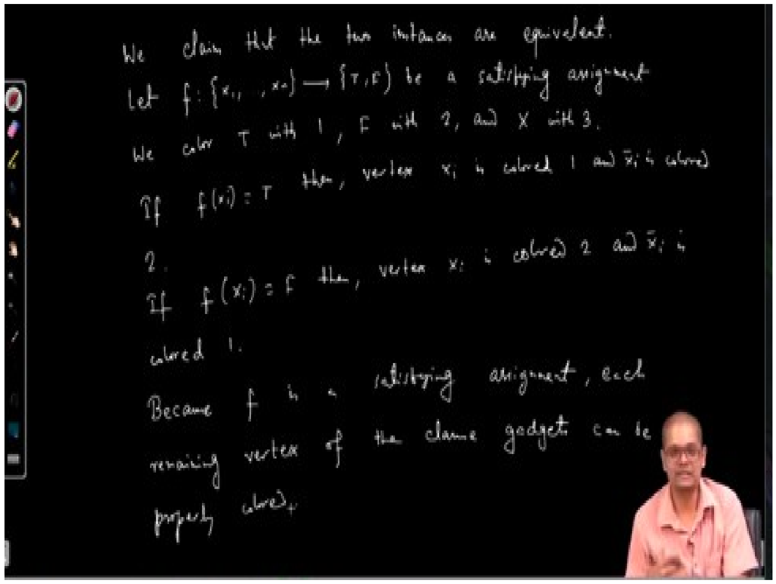

Now from so we claim that the two instances are equivalent. So, for that it is clear but let us formally prove it let f from $x_1 \ldots, x_n$ true and false be a satisfying assignment. So, we coloured T with one the vertex T with one vertex F with two and vertex X with 3. Now if $f(x_1)$ is true then vertex x_1 is coloured one and $\bar{x}_1$ is coloured 2. Similarly, if f of you can write $f(x_i)$ is false then vertex x_i is coloured 2 and $\bar{x}_i$ is coloured 1.

Now because this is a satisfying assignment all class gadgets can be coloured can be properly coloured. So, because if is a satisfying assignment that means for each clause l_1 or l_2 or l_3 there exist at least one literal which is set to true. Because f is a satisfying assignment each remaining vertex of the clause gadgets can be properly coloured. So, hence there exist a proper colouring, hence G has proper 3 colouring.

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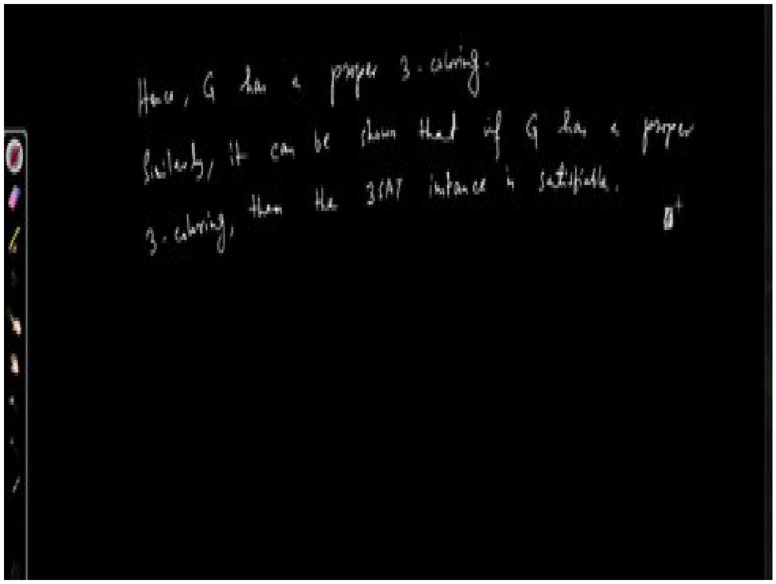

The same can be shown for the reverse direction also. So, similarly can be shown easily; that if G has a proper 3 colouring, then the 3SAT clause, 3SAT formula, 3SAT instance is satisfiable. Why? Because if the colouring is proper that means in each clause gadget l_1 or l_2 or l_3 at least one of the clause must be set to true. So, from where from there it follows that the 3SAT instance is satisfiable. So, this concludes the proof of the equivalence of the instance. So, we will stop here for today.