Selected Topics in Algorithm
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 35
NP – Completeness of Vertex Cover and Clique

Welcome we have been seeing we have been seeing some NP completeness proofs and NP complete reductions. So, let us continue that.

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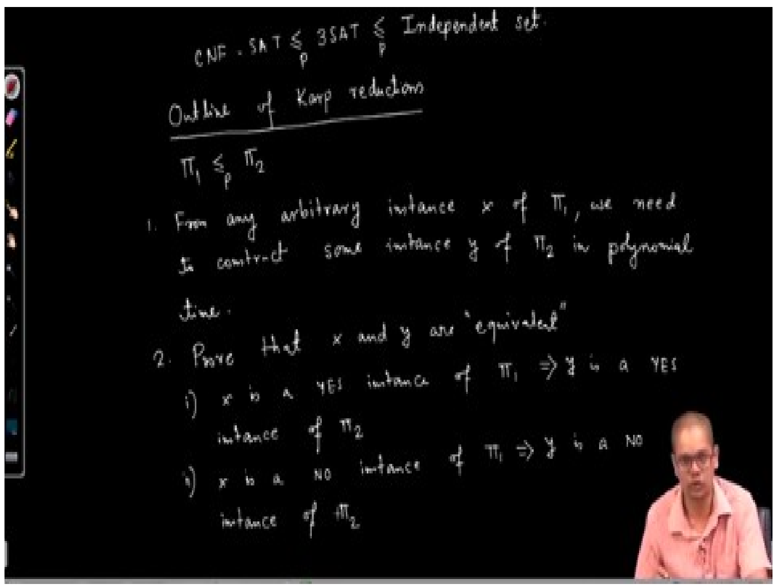

So, till now we have seen that no CNF-SAT reduces in polynomial time into 3SAT and 3SAT also reduces in polynomial time and these are all Karp reduction into independent set. So, what are the main steps for showing reductions? So, outline of reductions, see we are reducing from a problem Π_1 to Π_2 then first we need to from any arbitrary instance x of Π_1 we need to construct some instance y of Π_2 in polynomial time.

And then we need to show that x and y are equivalent prove that x and y are equivalent that involves proving two things first is x is a YES instance of Π_1 implies that y is YES instance of pi 2, that is 1 and the same holds for no this x is a NO instance of Π_1 implies that y is a NO instance of Π_2. So, this is the basic framework basic things for showing a polynomial time Karp reduction.

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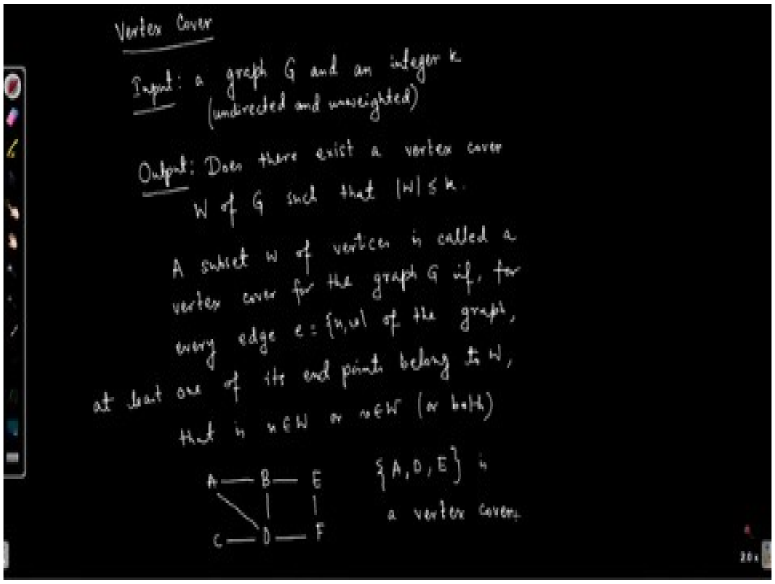

So, in today's class let us prove some more such reductions to get used to it. So, today our fast reduction is the vertex cover problem to show that vertex cover problem is NP complete. So, what is vertex cover problem? So, input is a graph G and an integer key whenever we say graph if not mentioned otherwise it should be understood that the graph is undirected and unweighted. So, these are the two inputs, output the decision problem.

So, if the output is either yes or no so, does there exist vertex covered W of G such that cardinality W is at most A. What is vertex covered? Subset W of vertices is called a vertex covered for the graph G if for every edge $e=\{u,v\}$ of the graph. One of its endpoints at least one of its endpoint belong to W, that is u belongs to W or v belongs to W of course or both. So, let us see an example graph A B C D E F so let us look at this graph.

So, this set of vertices A D and E forms a vertex cover all the edges are covered. For every word for every edge at least one of its endpoint belongs to this set. So, this is a vertex covered and this problem is called the vertex cover problem where the goal is to find if there exist a vertex cover of size at most k.

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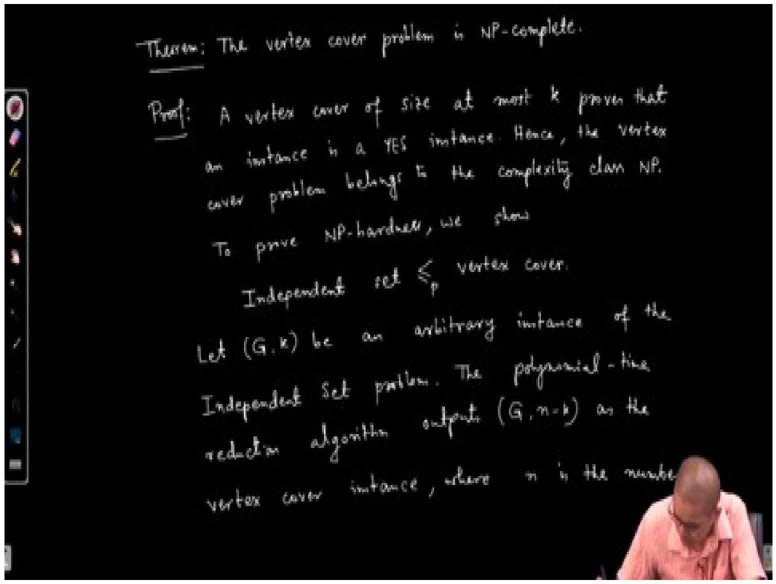

Vertex cover problem is NP complete, proof recall for a problem to be NP complete it needs to satisfy two things one is membership in NP another is NP hardness. Membership in NP means that for YES instances there should exist a polynomial size certificate which can be easily verified to be a proof for the easiness of the instance. So, for the vertex cover problem a psi a vertex cover of size at most came acts as a proof or a short certificate to convince anyone to that the instance is indeed a YES instance.

So, let me write a vertex cover of size at most k proves that an instance is a YES instance. Hence the vertex cover problem belongs to the complexity class NP. So, that is the first part membership in NP now to prove NP hardness we show independent set polynomial time Karp reduces. So, this sort of reductions are called many to one reductions polynomial time many to one reduces to vertex covered.

So, for that let G, k be an arbitrary instance of the independent set problem. We or the polynomial time reduction algorithm which is of course an algorithm which takes an independent set instance as input and outputs a vertex cover instance the polynomial time reduction algorithm outputs G, n - k as the vertex cover instance, where n is the number of vertices in the graph G.

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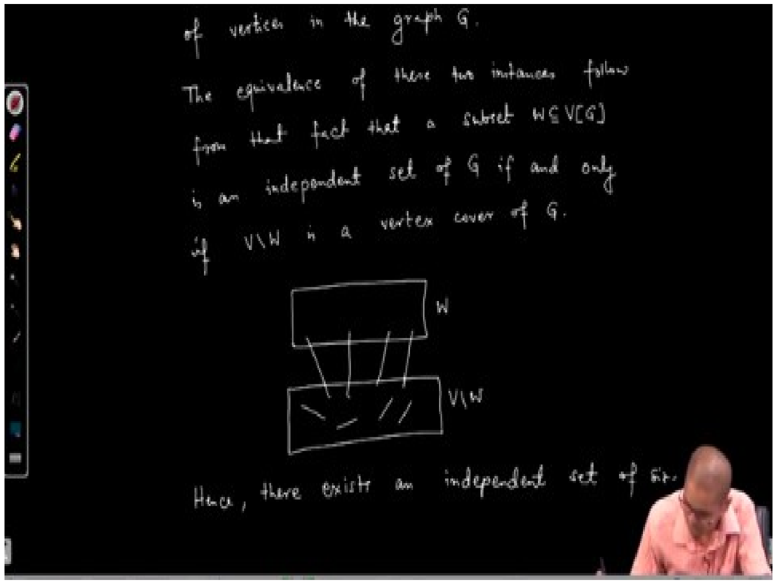

The equivalence now we need to show equivalence of these two instances. The equivalence of these two instances follow from the fact that subset W subset of the vertex set of G is an independent set of G if and only if the rest of the vertices V - W is a vertex covered of G. Let us see why let us prove this pictorially so suppose this is a subset of vertices W and this is the remaining set of vertices V – W and this if W is an independent set.

That means there cannot be any edge completely contained within W. So, all the edges are should be contained entirely in V - W that means here like this or they should be like cross edges then that means that V - W is a vertex cover all the edges has at least one end point in V - W simply because W is an independent set.

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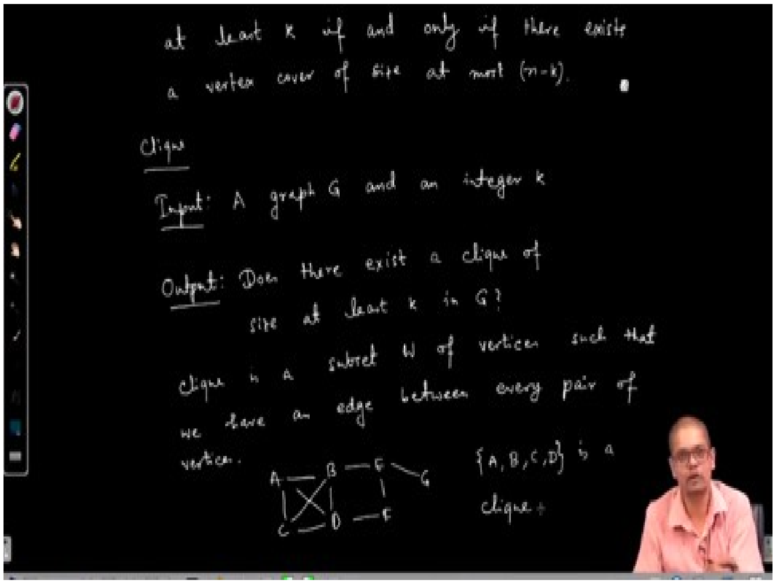

Hence, they are exists an independent set of size at least k if and only if there exists a vertex cover of size at most n - k which concludes the proof. Hence these two instances are equivalent. These sort of an easy reduction there is not so much sophisticated gadget involved. Another this sort of easy reduction is clique, so what is the clique problem? Input is a graph G again if not mentioned otherwise this graph is unweighted and undirected.

So, that is the kind of graph is then input a graph G and an integer key output does there exist clique of size at least k in G so, what is the clique? Clique is subset W of vertices such that we have an edge between every pair of vertices. So, again let us take an example of a graph A B C D E F G. So, in this graph A B C D E forms a clique remember it should be there A B C D is a clique because between every pair of vertices we have an edge.

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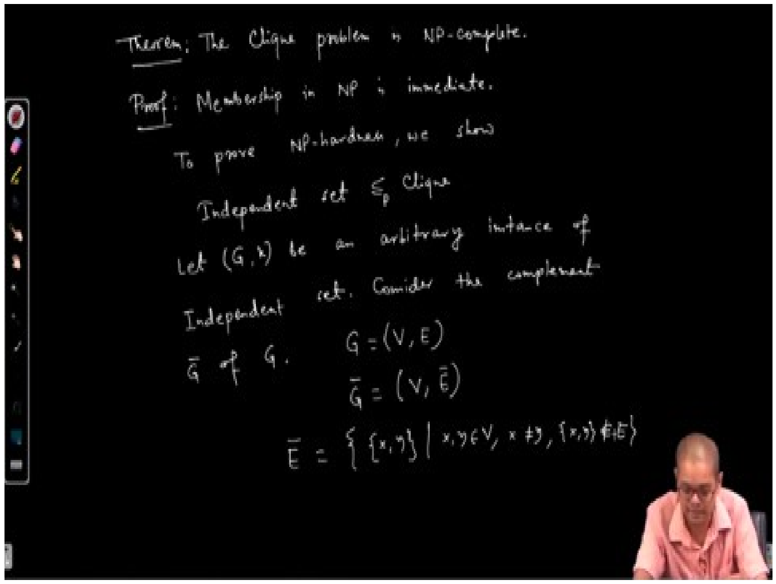

Now we show that clique is also NP complete the clique problem is NP complete, proof. Again, first we need to show memberships and membership is sort of obvious because you any clique of size at most k can act as a proof for easiness. So, membership in NP is immediate to prove NP hardness we show independent set polynomial time many-to-one reduces to clique. So, let G, k be an arbitrary instance of independent set consider the complement graph complement G bar of G.

What is complement? So, if G is (V, E) $\bar{G}$ is $(V, \bar{E})$, now what is $\bar{E}$? $\bar{E}$ is x, y belongs to x, y such that $x, y \in V, x \neq y$ and this edge $\{x, y\}$ does not belong to E. So, for all pairs of vertices if there is no edge, I put an edge. So, that is $\bar{E}$.

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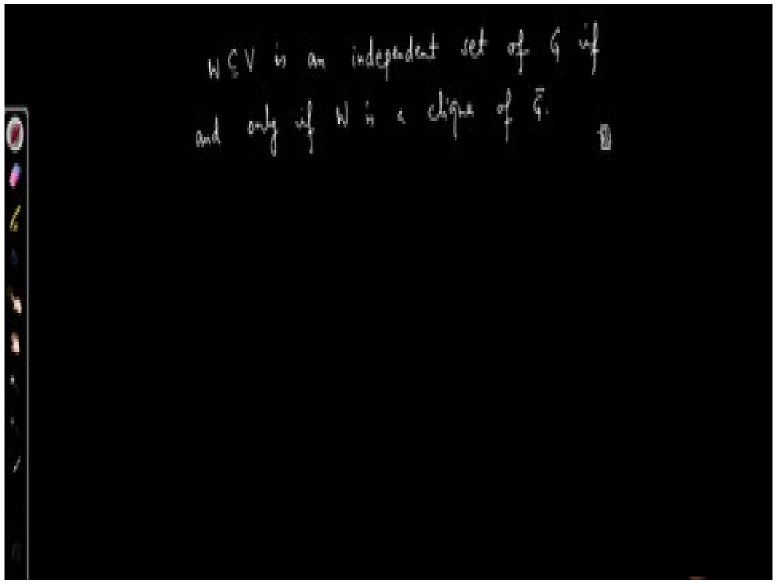

And now this is easy to see that if W subset of V is an independent set of G if and only if W is a clique of $\bar{G}$. Hence there exist an independent set of G of size at least k if and only if there exist a clique of size at least k in $\bar{G}$ which concludes the equivalence of these two instances which shows that clique problem is NP hard. So, in the next class we will see some more non trivial reductions, thank you.