Selected Topics in Algorithm
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Lecture - 34
NP - Completeness of Independent Set

Welcome, in the last class we have seen the reduction of CNF-SAT to 3SAT, in this class we will see some more reductions.

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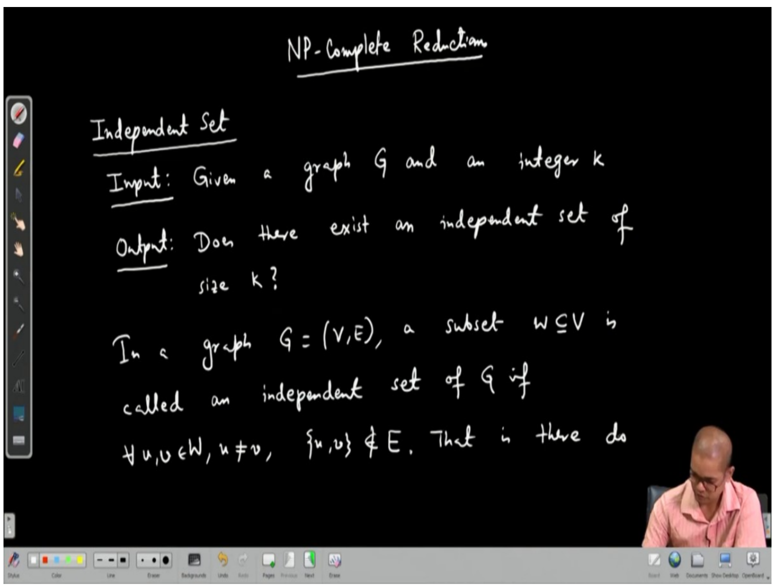

So, NP complete reductions. So, in our last reduction from CNF-SAT to 3SAT you know both are like satisfiability problem both are problems of same flavor. Now today we will see a graph problem to be NP hard NP complete and we will reduce it from 3SAT. So, for that let me introduce the problem called independent set, given graph G and an integer k, output does there exists and independent set of size k, that is the question.

So, what is an independent set? In a graph $G=(V,E)$, a subset W of V is called an independent set of G if for all $u,v\in W, u\neq v$ there is no edge between u and v. That is their does not exist an edge between any two vertices of G.

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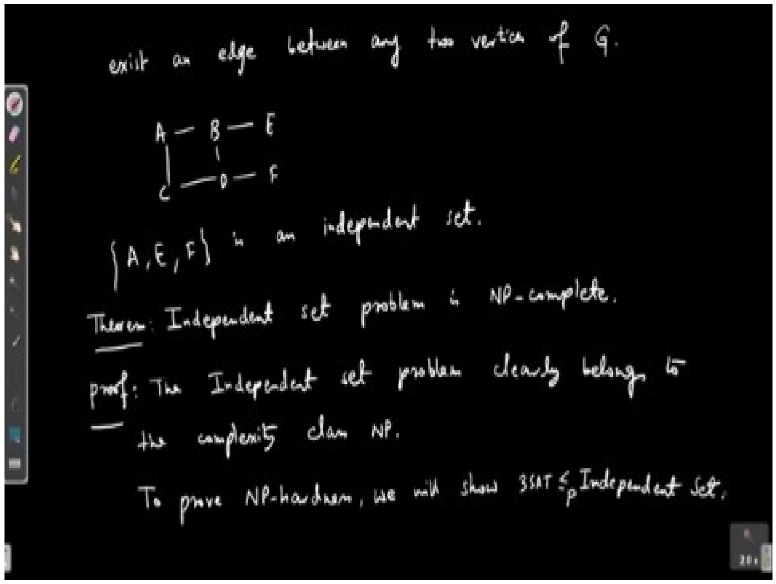

So, let us take a concrete example A B C D E F, so A E and F this set of vertices forms are independent set because there is no edge on two vertices on it. So, is the problem of finding this is called a max independent set, this independent set problem is it NP complete? Yes, so this is our result today, independent set problem is NP complete, proof, again what is NP completeness it has two parts one is membership in NP another is NP hardness.

So, the independent set problem clearly belongs to the complexity class NP. Why? Because then independent set if it is a YES instance that means there exists are independent set of size k. Now that independent set itself can be used as a certificate, it is easy to verify that that set as k vertices and there is no edge between any two vertices and that is it. So, this membership in NP is clear. So, to prove NP hardness we will show 3SAT polynomial time Karp reduces to independent set.

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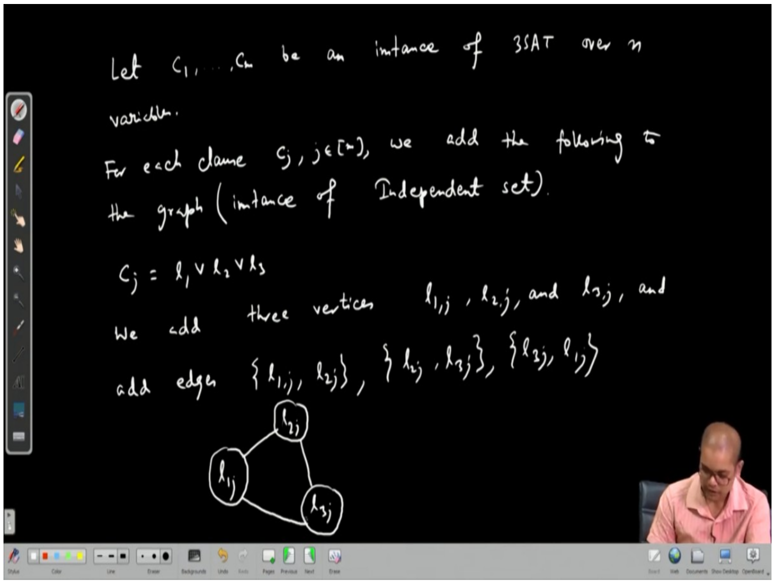

So, for that we need to convert an instance of every instance of 3SAT to an equivalent instance of independent set. So, let $C_1, \ldots, C_m$ be an instance of 3SAT over n variables and now we will construct a graph. So, for each clause $C_j, j \in [m]$ we add the following to the graph which is an instance of independent set. So, it has three literals so C_j is $l_1 \vee l_2 \vee l_3$, then we add three vertices with labels three vertices l_{1j}, l_{2j}, and l_{3j} and add edges l_{1j}, l_{2j}, this all three edges l_{2j}.

Let me not write the comma there is may be confusing l_{3j} and l_{3j}, l_{1j}. So, for each clause we add these triangles so basically pictorial it looks like l_{1j} is one vertex, l_{2j} this is another vertex and l_{3j} this another vertex and we add these three edges, so for each clause we do this.

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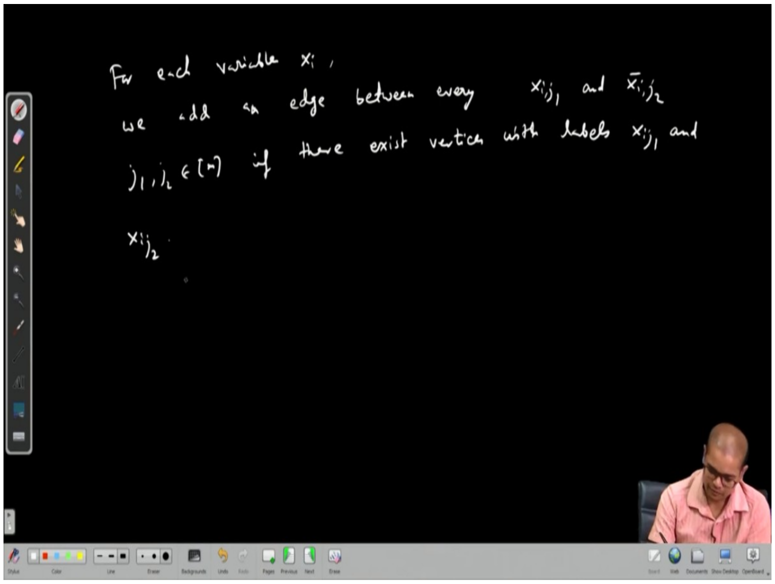

And now for each variable no x_i we add an edge, we are not introducing any new vertex we are just adding edges, add an edge between every you know x_{ij_1} and $\bar{x}_{ij_2}$ where $j_1, j_2 \in [m]$ if there exists vertices with labels x_{ij_1} and x_{ij_2}. So, let us take an example that way it will be clear. So, suppose I have a 3SAT formula, let us go to next page.

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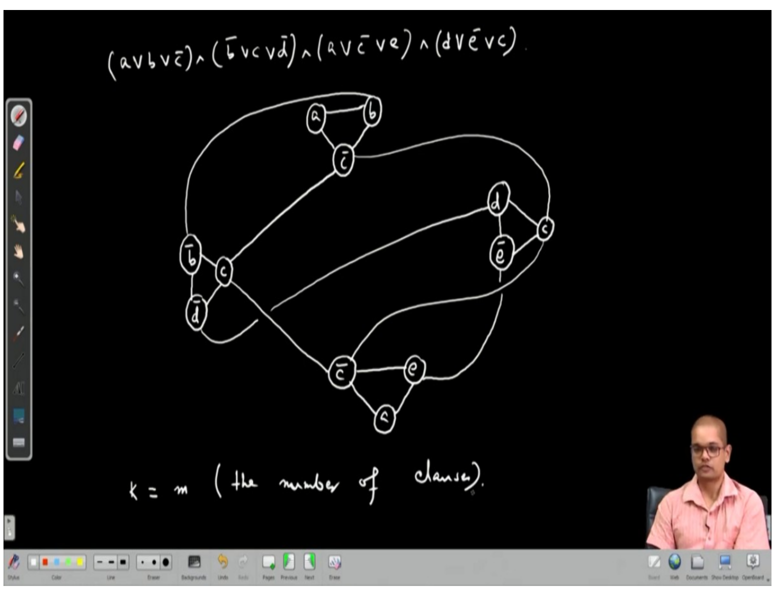

Suppose I have a 3SAT formula the clauses are maybe $a \vee b \vee \bar{c}$ and you know $\bar{b} \vee c \vee \bar{d}$ and suppose it is $a \vee \bar{c} \vee e$ and suppose it is $d \vee \bar{e} \vee c$. Suppose this is the formula then what is my constructed three constructed instance of independent set, for each clause I first create a triangle.

So, the first clause is $a \vee b \vee \bar{c}$, so I create three vertices a b and c bar with these labels and these are for this clause only and add them with edges.

So, this is a triangle similarly, $\bar{b} \vee c \vee \bar{d}$, $a \vee \bar{c} \vee e$ so the triangles and we have $d \vee \bar{e} \vee c$ so for each clause I create these triangles. Now I add edges between a literal and its complement and that is the idea that independent set should not pick should pick at most one of them it should not pick both of them. So, c and $\bar{c}$, so these are the edges for c let us see for a appears only as positive here is b and $\bar{b}$, here is d and $\bar{d}$ and here is e and $\bar{e}$.

I think these are the things and we set k to be m, the number of clauses. So, I have a graph and a k.

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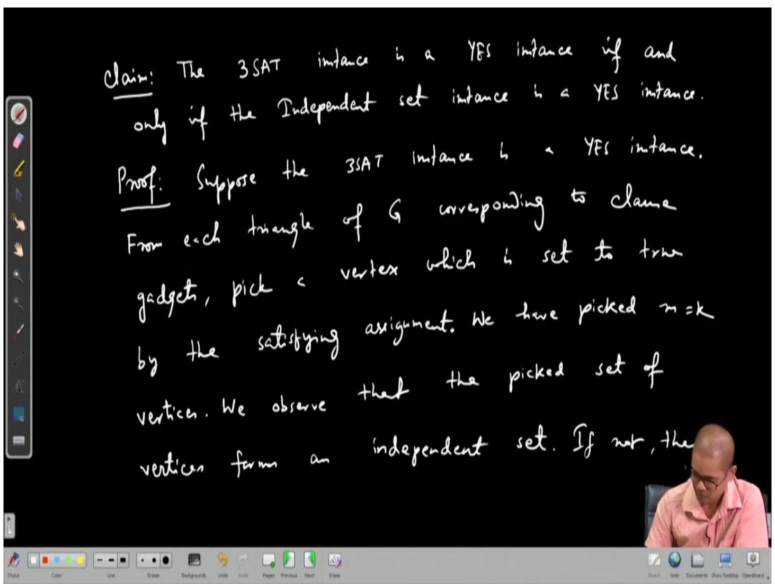

Now we claim the 3SAT instance is a YES instance if and only if the independent set instance is a YES instance. Proof, so suppose the 3SAT instance is a YES instance. Now that means each clause has a satisfying is each clause has a variable which is satisfied as a literal which is satisfied. So, these are called each of these boxes these triangles it is called clause gadget, it is like for each clause we are creating a structure in the instance these are clause gadgets.

So, from each triangle of G corresponding to clause gadgets pick a variable pick a vertex which is set to true by the satisfying assignment you know because the 3SAT clause is satisfied by the assignment each clause contains a literal which is set to true. So, pick that corresponding literal so and because there are m clauses we have picked m vertices and m is k, so we have picked m which is equal to k vertices.

Now we claim, we observe that the picked set of vertices forms and independent set. So, suppose not if not if it is not then that means there is an edge but ages are between a variable and its complement.

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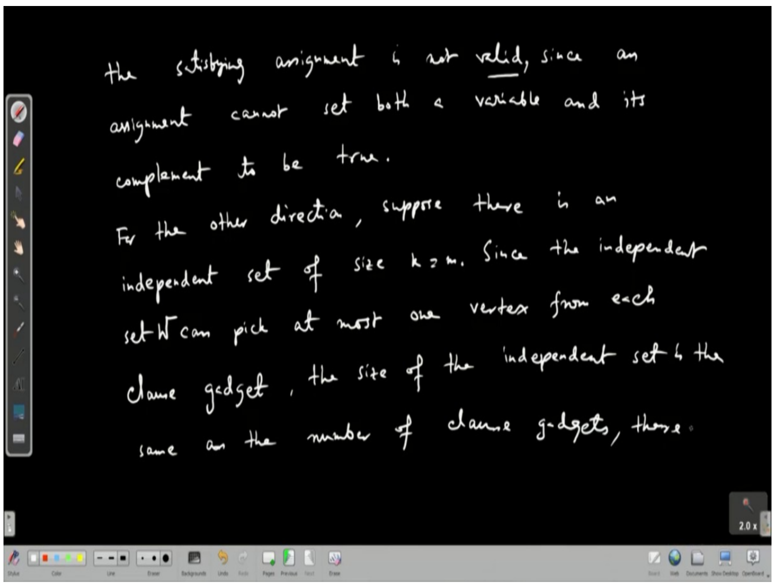

So, if not then the assignment the satisfying assignment is not valid indeed whichever vertex, we picked that is set to true by the satisfying assignment. Now an assignment cannot set a variable and its complement to be true. Let me write since an assignment cannot set both a variable and it is complement to be true. So, hence if there exist a satisfying assignment then there exists an independent set of size m.

How about the other direction? So, for the other direction recall you need to prove equivalence. That means the 3SAT instance is satisfiable if and only if the independent set is a YES instance there exists an independent set of size m. So, for the other direction suppose they are is an independent set of size k = m. Now that independent set can pick at most one vertex promise each clause gadget and there are m clause gadgets.

So, since the independent set can pick at most one vertex from each clause gadget and the size of the independent set, we write this. The size of the independent set is the same as the number of clause gadgets the independent set, let us call this independent set W.

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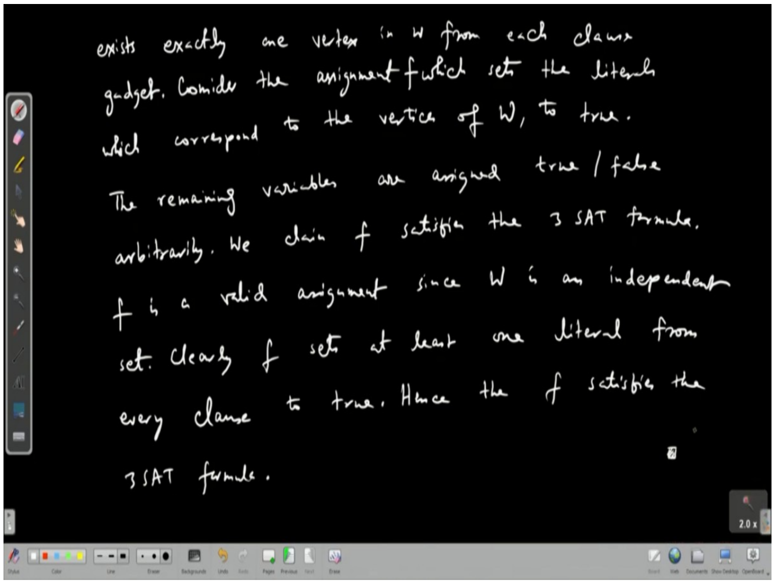

There exists exactly one vertex in W from each clause gadget. Now consider the assignment which sets the literals which corresponds to the vertices of W, to true that means whichever vertices are in W the corresponding literals set it to true and the remaining variables are assigned are true or false arbitrarily, we claim. So, let us call this assignment, f so, we claim f satisfies the 3SAT formula.

First of all, f is a valid assignment it does not set both a variable and its complement to be true. So, f is a valid assignment since W is an independent set because we have added edges between every variable and its complement and f is setting only the variables in W to be true. So, f cannot set a both a variable and its complement to be true if it does that means W has both of that variable and the both the vertices corresponding to the variable and its complement.

But then W has an edge it will not be an independent set that means. Hence f is a valid assignment and of course if sets every f sets at least one variable in every clause to be true and clearly f sets that is by the construction of, f sets at least one literal from every clause to true. Hence the 3SAT formula hence f satisfies the 3SAT formula. Hence the 3SAT instance is a YES instance which establishes the equivalence of this these two instances which shows that it is a valid reduction.

And hence the independent set problem is NP hard. So, the next class we will see some more examples of NP completeness, so let us stop here.