Selected Topics in Algorithm
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Lecture 30
Counting Independent Sets of a Graph (Continued)

Welcome in the last class, we have started seeing Markov Chain Monte Carlo method and using this method how we can design FPRAS for Counting the Number of Independent Sets of a Graph.

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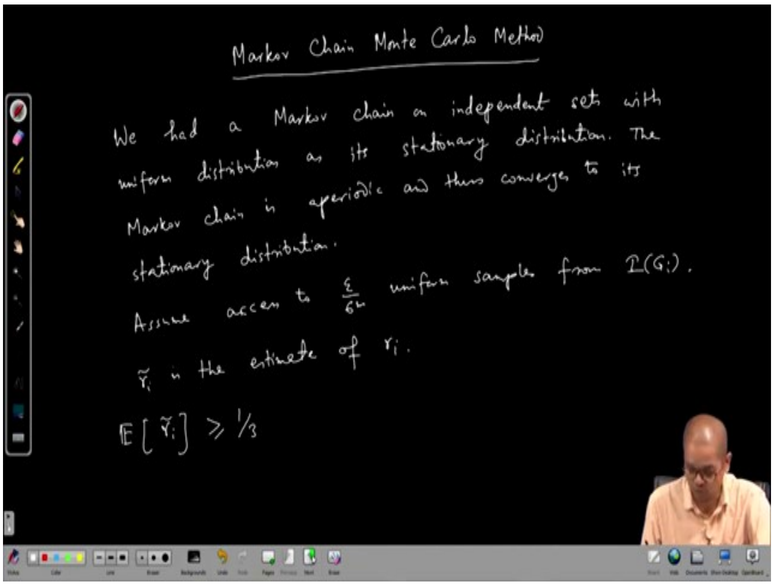

So, let us continue that Markov Chain Monte Carlo method. So, we had a Markov Chain on independent sets with uniform distribution as it is stationary distribution. And the mark option is aperiodic the Markov Chain was aperiodic of course finite and hence thus converges to it is stationary distribution. So, it converges to it is stationary distribution So, using this framework assuming mixing time is small.

We assume access to $\frac{\epsilon}{6m}$ uniform samples from the set of all independent sets. And if $\tilde{r}_i$ is the estimate of r_i then in the last class, we have seen that $\tilde{r}_i$ is expectation of $\tilde{r}_i$ is greater than equal to one third and this makes the Chernoff bound applicable. Because if excitation is very close to 0 then the Chernoff bound the inequality or the bound that it gives is not very useful.

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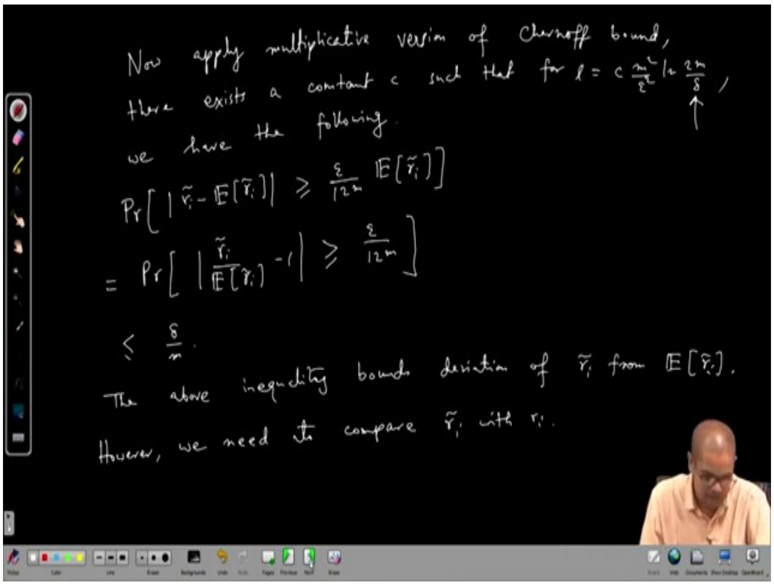

So, let us continue that now applying multiplicative version of Chernoff bound there exists a constant c, such that for $l = \frac{cm^2}{\epsilon^2}\ln\left(\frac{2m}{\delta}\right)$, we have the following. So, this is from again estimator theorem that we want to estimate $\tilde{r}_i$. So, probability that $\tilde{r}_i$ minus expectation of $\tilde{r}_i$ is greater than equal to $\frac{\epsilon}{12m}$ expectation of $\tilde{r}_i$.

This is probability that $\frac{\tilde{r}_i}{E[\tilde{r}_i]} - 1$ is greater than equal to $\frac{\epsilon}{12m}$, this is less than equal to $\frac{\delta}{m}$. Because here we have chosen we put $\ln\left(\frac{2m}{\delta}\right)$ that is why the error probability is $\frac{\delta}{m}$. So, the above inequality bounds deviation of $\tilde{r}_i$ from $E[\tilde{r}_i]$ but we need to compare r_i and δ.

Because our goal is to compute r_i, estimate r_i, we write however we need to compare $\tilde{r}_i$ with r_i. But for that let us recall, we had already had some bound of deviation of expectation $\tilde{r}_i$ from r_i, let us recall what was that?

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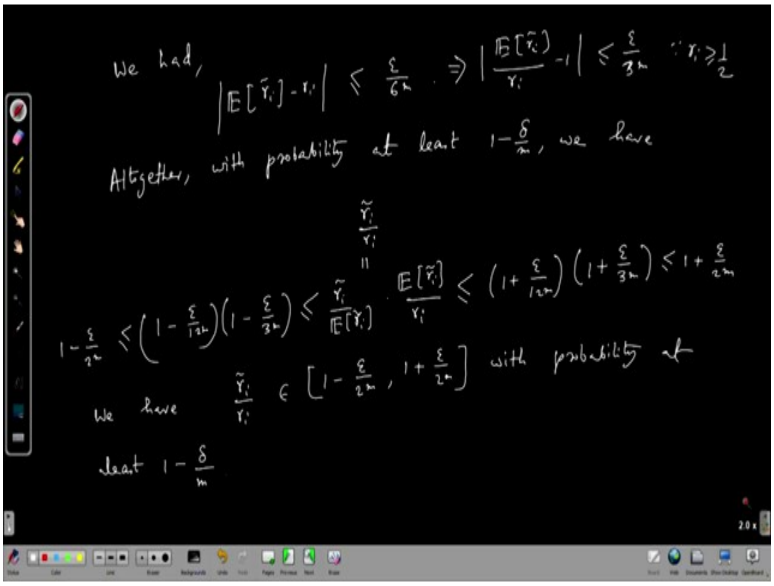

We had expectation of $\tilde{r}_i - r_i$ this is at most $\frac{\epsilon}{6m}$. So, this always holds and this hold with probability and $\tilde{r}_i$ is $\frac{\epsilon}{12m}$ close to r_i expectation of $\tilde{r}_i$ in multiplicative term with probability at least $\frac{\epsilon}{2m}$. So, with probability, so hence or all together with probability at least $1-\frac{\delta}{m}$. We have what is $\frac{\tilde{r}_i}{r_i} = \frac{\tilde{r}_i}{E[\tilde{r}_i]} \frac{E[\tilde{r}_i]}{r_i}$.

Now, let us bound in them $\frac{\tilde{r}_i}{E[\tilde{r}_i]}$ this is at most $1+\frac{\epsilon}{12m}$. So, this is less than equal to the first term is at most $1+\frac{\epsilon}{12m}$. The second term expectation of $\tilde{r}_i$ so, this is 6 m and from there we got that expectation of $\frac{\tilde{r}_i}{r_i} - 1$ this is less than equal to $\frac{\epsilon}{3m}$, here we use that $r_i \geq 1/2$.

So, expectation of $\frac{\tilde{r}_i}{r_i}$ is at most $1+\frac{\epsilon}{3m}$ on the lower side, the first term is at least $1-\frac{\epsilon}{12m}$. The second term expectation of $\frac{\tilde{r}_i}{r_i}$ this is at least $1-\frac{\epsilon}{3m}$. So, then this is, but this is less than equal to $1-\frac{\epsilon}{12m}+\frac{\epsilon}{3m}$. So, this is at least $1-\frac{\epsilon}{2m}$ and this is less than equal to $1+\frac{\epsilon}{2m}$, ignoring ϵ^2 terms.

So, each so we have $\frac{\widetilde{r_i}}{r_i}$ belongs to this interval $[1-\frac{\epsilon}{2m}, 1+\frac{\epsilon}{2m}]$ with probability at least $1-\frac{\delta}{m}$.

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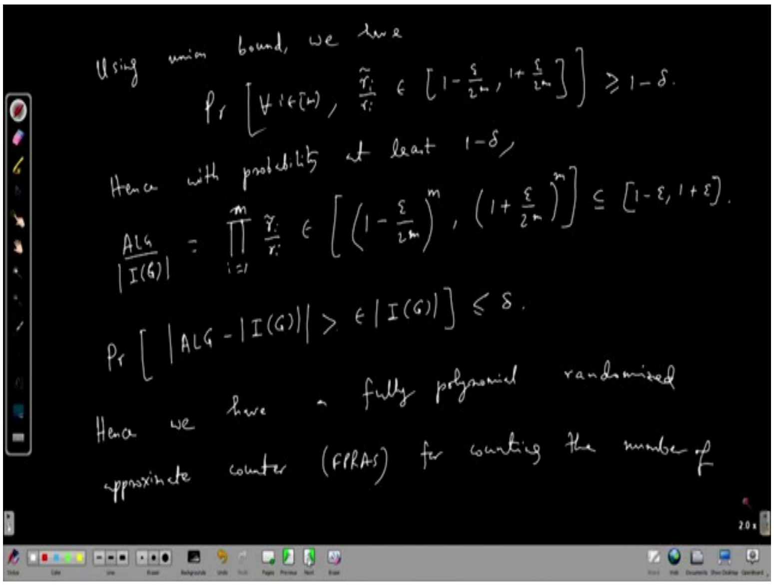

And now using union bound, we have $\frac{\widetilde{r_i}}{r_i}$ for all i in m this particular event. This value is in between or this belongs to $\left[1-\frac{\epsilon}{2m}, 1+\frac{\epsilon}{2m}\right]$ this for all $i \in [m]$ probability of this is greater than equal to $1-\delta$ this is from union bound. So, hence with probability at least $1-\delta$ we have ALG by $I(G)$ this is product over $\frac{\widetilde{r_i}}{r_i}$ i equal to 1 to m.

This is this product belongs to $1-\frac{\epsilon}{2m}$ each of the term is greater than equal to $\frac{\epsilon}{2m}$. So, this is this product is greater than equal to $\left[\left(1-\frac{\epsilon}{2m}\right)^m, \left(1+\frac{\epsilon}{2m}\right)^m\right]$. This is contained in $[1-\epsilon, 1+\epsilon]$. Hence ALG hence probability that ALG minus cardinality of independent sets.

Number of independent sets in G this is more than ϵ times cardinality of number of independent sets in G this is less than equal to δ. Hence, we have fully polynomial time FPRAS is fully polynomial randomized approximate counter FPRAS fully polynomial a randomized approximate counter for counting the number of independent sets in a graph.

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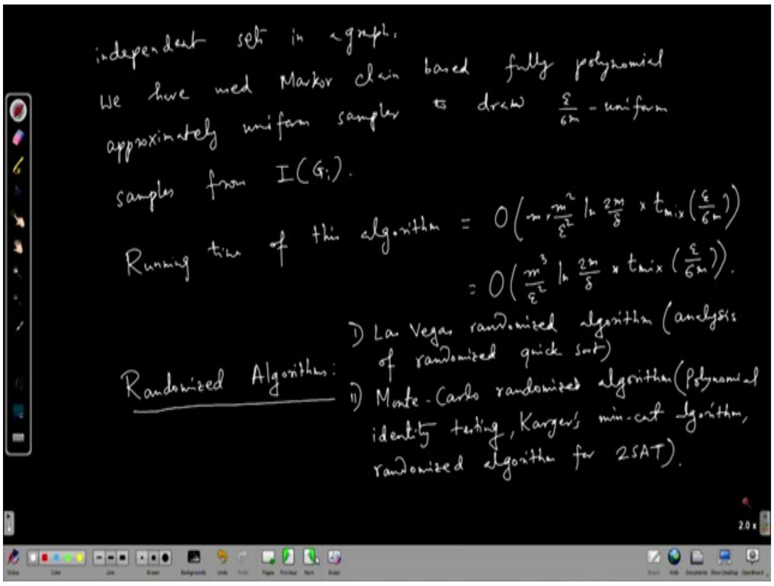

And we have used Markov Chain based fully polynomial approximately uniform sampler to draw epsilon by 6m uniform samples from $I(G_i)$. So, what is the running time of this algorithm? Running time of this algorithm is big O of so, to sample each to estimate each r_i we are drawing this many samples $\frac{cm^2}{\epsilon^2}\ln\left(\frac{2m}{\delta}\right)$ samples that is for each $r\,i$ and we need to compute estimate $r_1,\ldots,r_m$.

So, there are m many this m times this and to draw one sample, we need t_{mix} that is the mixing time of the Markov Chain on the independent sets and this is $\frac{\epsilon}{6m}$. This is $O\left(\frac{m^3}{\epsilon^2}\ln\left(\frac{2m}{\delta}\right)t_{mix}\left(\frac{\epsilon}{6m}\right)\right)$. So, let us briefly recall what is the sort of or technique that we have seen in randomized algorithms parts? So, this concludes the randomized algorithms part.

So, in the randomized algorithm parts, we see some two types of randomized algorithms. One is Las Vegas randomized algorithm and here we have seen analysis of quick sort analysis of randomized quick sort. Then we have seen couple of Monte Carlo algorithms Monte Carlo randomized algorithms. Here we have seen polynomial identity testing and then Karger's min cut algorithm and randomized algorithm for 2SAT.

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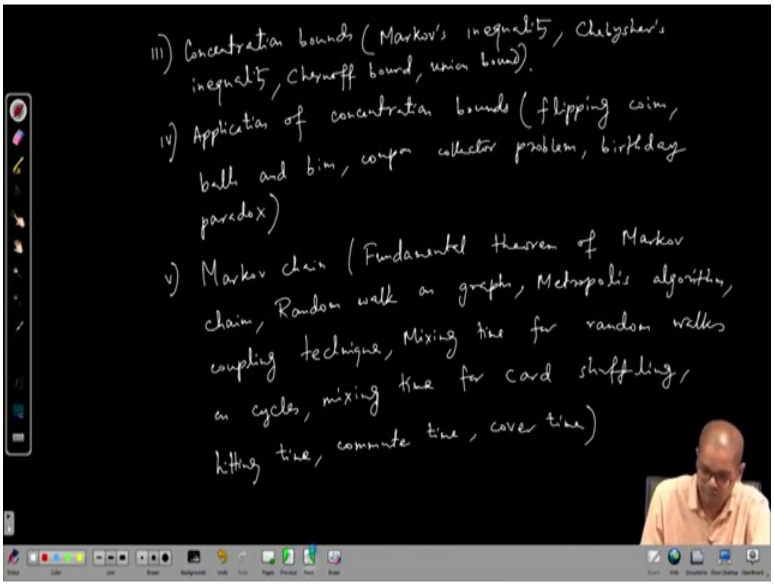

Next, we see we had seen various concentration bounds. Here comes Markov's inequality then Chebyshev's inequality then Chernoff bound and union bound. Then we see applications of concentration bounds. Here we have seen flipping coins then balls and bins then coupon collector problem, birthday paradox and so on. Then we move on to what is called Markov Chain, in Markov Chain we have studied fundamental theorem of Markov Chain.

And then many examples Markov Chain on circles, random walk on graph, metropolis algorithm. Then we have seen coupling technique then we have seen mixing time for some special Markov Chain mixing time for a random walk on cycles. Then mixing time for card shuffling and then other concepts like hitting time, commute time, cover time. And using this we have seen a randomized algorithm to discover the graph what is the cover time of random work on graph?

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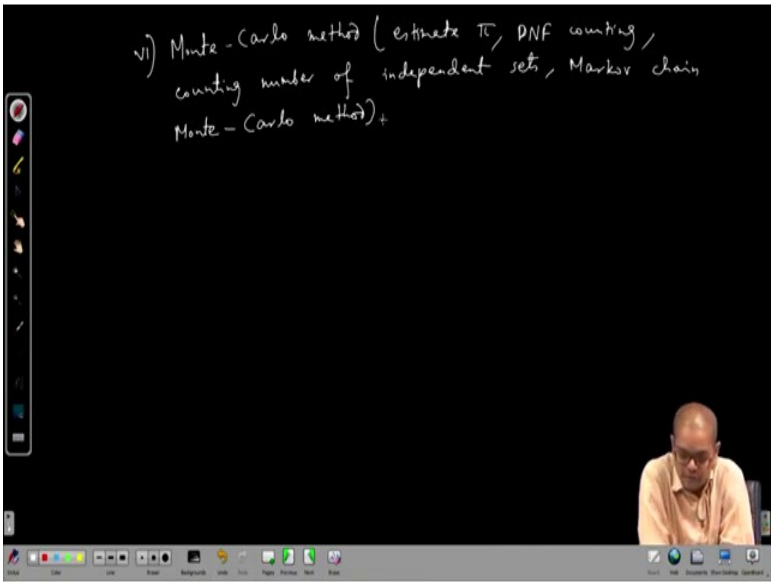

Then finally, we have seen Monte Carlo method, here we have seen estimate π, DNF counting and number of independent sets and here we have seen Markov Chain Monte Carlo method. So, this concludes the second part of the course which is like randomized algorithm. So, in from the next class we will study intractability namely NP-completeness and so on. Thank you.