Selected ToPics in Algorithms
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Module No # 05
Lecture No # 25
Shuffling Cards

Thank you welcome we have been seeing the application of Markov Chain and particular random walk on graphs. So in the last class we have started studying the example of shuffling cards.

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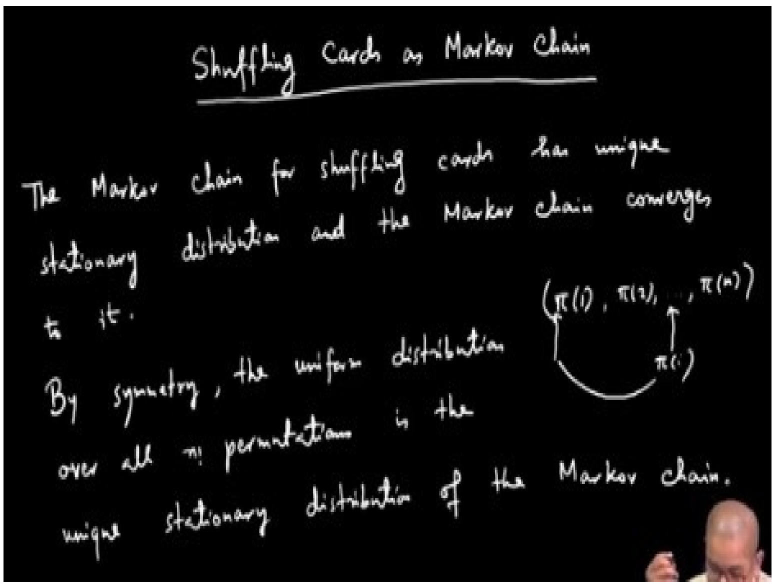

When we observe that the Markov Chain for shuffling cards as unique stationary distribution and the Markov Chain converges to it and what was the Markov Chain? So let us recall this is a permutation $\pi_1, \pi_2, \ldots, \pi_n$ and we pick a random card say π_i and put it at the top here, so basically shift it here. And by symmetry; so what is the unique stationary distribution? By symmetry the uniform distribution overall n factorial permutations is the unique stationary distribution of the Markov Chain.

Now hence if the mixing time is small after that many number of steps the distribution of X_t the t-th state of the Markov Chain will be very close to uniform distribution so that is the idea.

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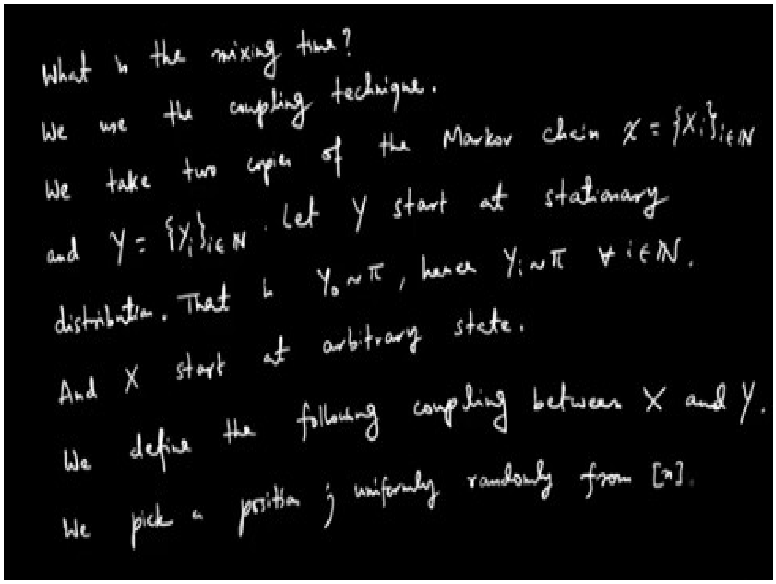

So now what is the mixing time? How many times we need to shuffle the card? Again let us the idea the technique is similar to Markov Chain on the cycle so we use coupling technique. Basically we take 2 copies of the Markov Chain $X=(X_i)_{i\in \mathbb{N}}$ and $Y=(Y_i)_{i\in \mathbb{N}}$. As usual one Markov Chain starts at stationary distribution hence it will remain at stationary distribution. So let Y start at stationary distribution that is Y_0 is distributed according to the stationary distribution.

Hence Y_i is also distributed according to π for all $i \in \mathbb{N}$; and X_t and X start at arbitrary state. So we define the coupling now; we define the following coupling between X and Y what is that coupling? That Pick up position j uniformly randomly from n.

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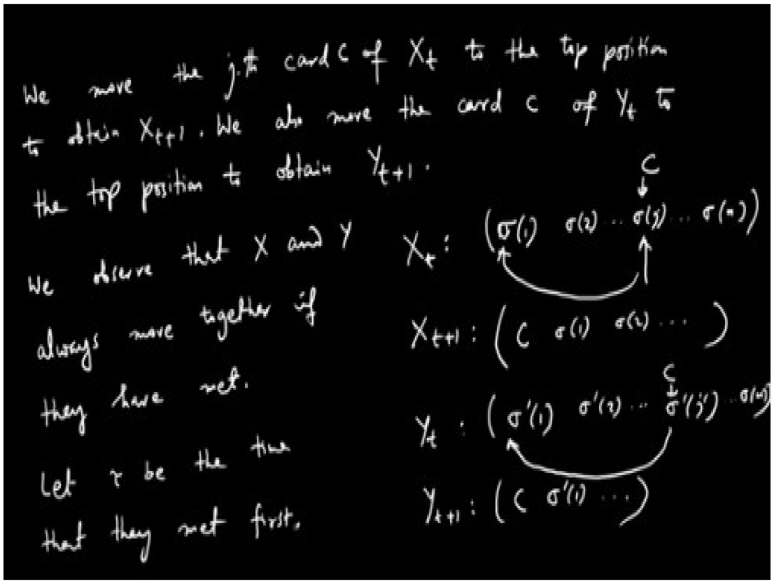

We move the jth card of X_t is a computation of jth card c of X_t to the top position to obtain X_{t+1} that is how this this Markov Chain is defined; you Pick a card uniform layer random jth cards and put it in the top. Next we also move for Y also I need to perform the same task but instead of Picking a position j uniformly random we in Y also we Pick that cards here and move at the top.

So anyone who is looking only at the Y process it is also one card from n cards is picked uniformly at random and put in this in the top place. We also move the card c of Y_t to the top position to obtain Y_{t+1}. So what is the process let us pictorially draw it so here is X_t how it look like it is a permutation let us call it sigma because you using π we typically denote the stationary distribution.

Suppose $\pi=(\sigma(1),\sigma(2),\dots,\sigma(j),\dots,\sigma(n))$ and one card one position is picked uniformly at random and moved at the top place. Suppose this is the card c then X_{t+1} is what this card c then $\sigma(1),\sigma(2)$ and rest. For why what you do is that Y_t it is some other $(\sigma'(1),\sigma'(2),\dots,\sigma'(j),\dots,\sigma'(n))$ here also the you move the cards here at the top so why t + 1 is c sigma prime 1 and so on.

So here also in both the cases you see that if you look at the individual Markov Chain they are following their right distribution in both X and Y if you look one process either X or Y. In each case one of the cards is speed uniformly at random and placed at the top. And of course this is when so this is followed when X and Y as not met. And of course if X and Y as met this is also the same so no problem to write this.

So what you observe is that? By this definition itself we observe that X and Y always move together if they have met. And on the distribution of Y is stationary distribution so let tau be the time that they made first. So when will X and Y will meet first of course when all of the cards have been seen.

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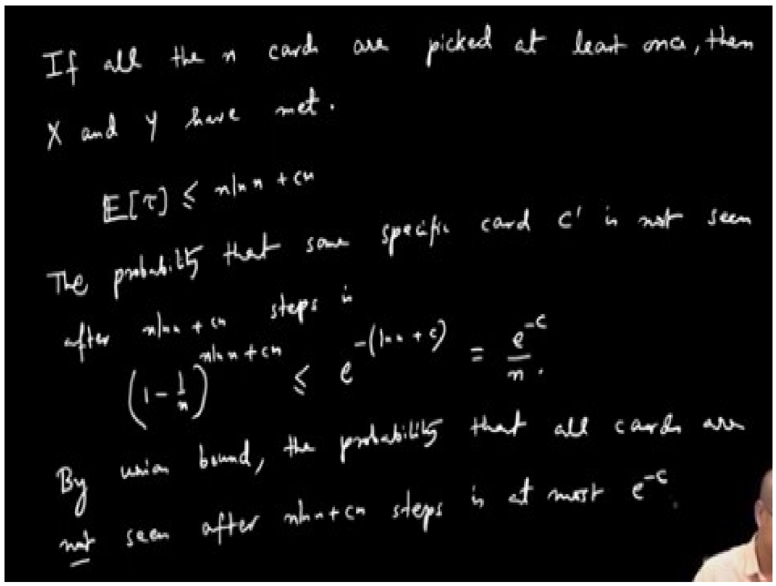

So if all the cards in the deck all the in cards are picked at least one then X and Y has met. So we have n cards and we are picking one card uniformly at random; in every iteration in every step. So how many steps in expectation are required to see all the cards, this is the classical coupon collector problem and we have seen that the expected time to see all the cards is less than equal to $n\ln n+c$ it is order in line in $n\ln n+c$.

So let us do it the probability that some specific card say c prime is not seen after $n\ln n+c$ steps is $1-\frac{1}{n}$ is the probability that it is not seen in one draw and I am spending $n\ln n+c$ steps. So this is $\left(1-\frac{1}{n}\right)^{n}$ this is less than equal to e^{-1} is $e^{-\ln n+c}$ this is $\frac{e^{-c}}{n}$. Hence by union bound the probability that all cards are not seen after n learning + c n draws or steps is at most e^{-c}. So hence to make this probability less than ϵ we should make c to be $\ln\left(\frac{1}{\epsilon}\right)$.

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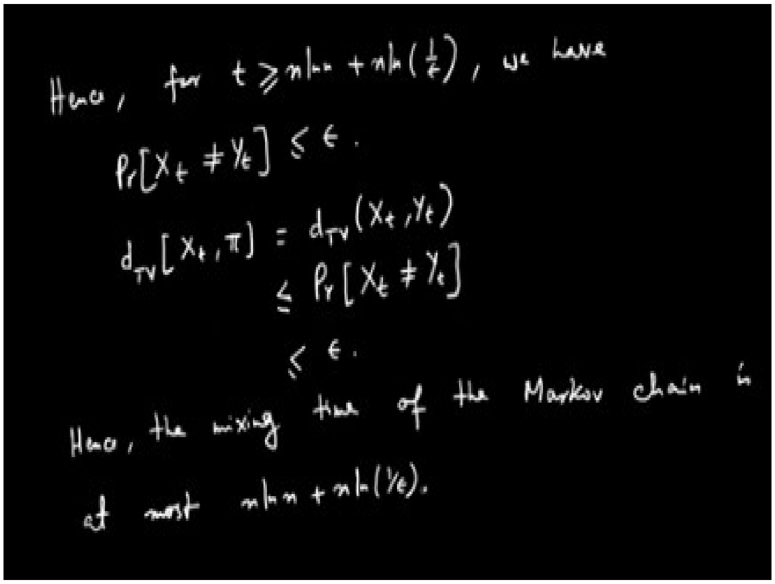

So hence for $t \geq n\ln n + n\ln\left(\frac{1}{\epsilon}\right)$ we have probability that $X_t \neq Y_t$ this is less than equal to epsilon. Now by coupling lemma we know that this is the total variation distance $d_{TV}[X_t, \pi]$ which is total variation distance between X_t and Y_t. Because distribution of Y_t is π and this is the less than equal to probability that X_t is not equal to Y_t this is less than equal to epsilon.

Hence the mixing time of the Markov Chain is at most $n\ln n + n\ln\left(\frac{1}{\epsilon}\right)$. So after this many steps all cards the state X_t is one of the permutations with probability one over n factorial minus at most ϵ. So this probability is very ϵ close to uniform distribution in total variation distance. So these are the some of the applications of random walk on Markov Chains.

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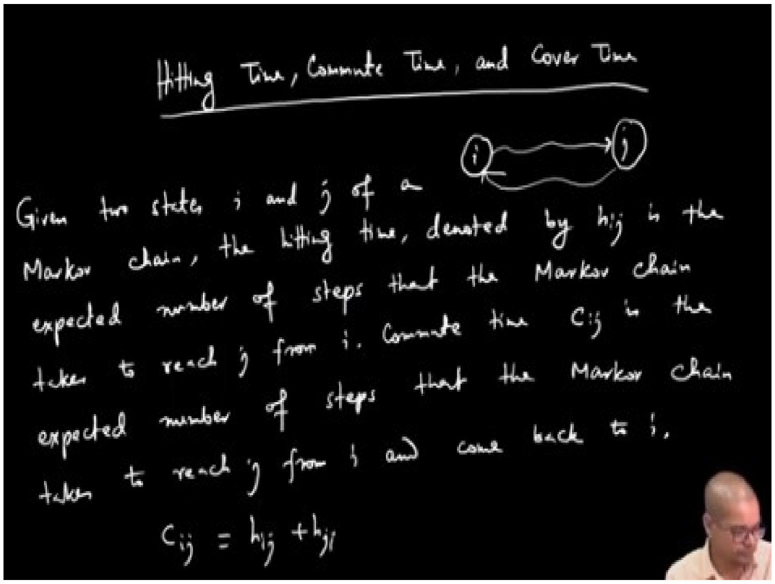

Next we see some more concepts of Markov Chain which are called hitting times, commute time and cover time. So it is like you know from state i what is the expected number of step to hit another state g that is sort of the hitting time? And commute time is from i how many how many steps in expectation need to reach j and come back to y that is commute time. And cover time is from a state how; what is the expected number of steps to reach all the states as at least one.

So what are them let us say write it formally given 2 states i and j of a Markov Chain the hitting time denoted by h_{ij} is the expected number of steps that the Markov Chain takes to read j from i. Commute time C_{ij} is the expected number of steps that the Markov Chain takes to read j from i and come back to i. So $C_{ij}=h_{ij}+h_{ji}$.

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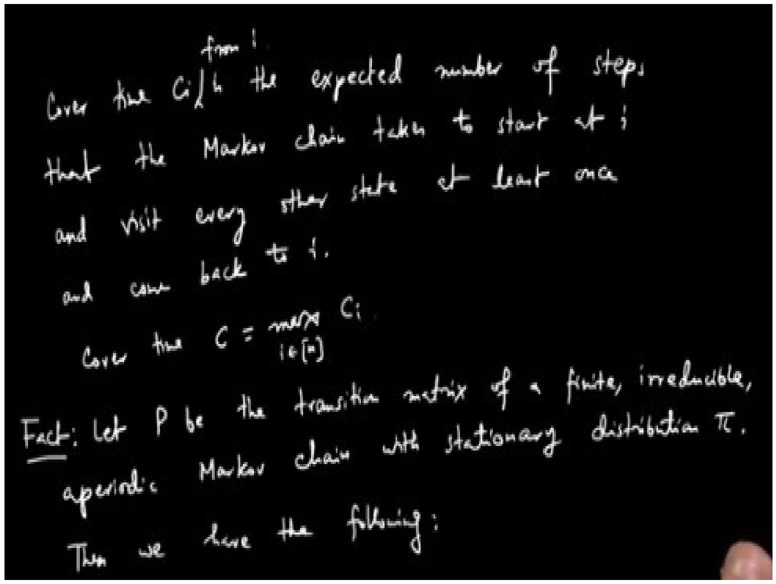

And cover time C_i is the expected number of steps that the Markov Chain takes to start at i and visit every other state at least once at least one and come back to i cover time C i from i; and cover times $C = max_{i \in [n]} C_i$ max of maximum of C_i. Now turns out that these quantities are related very closely related to the stationary distribution and will state some fact without proof fact. Let P be the stationary matrix the transition matrix of a finite irreducible aperiodic Markov Chain with stationary distribution Pi; then we have the following.

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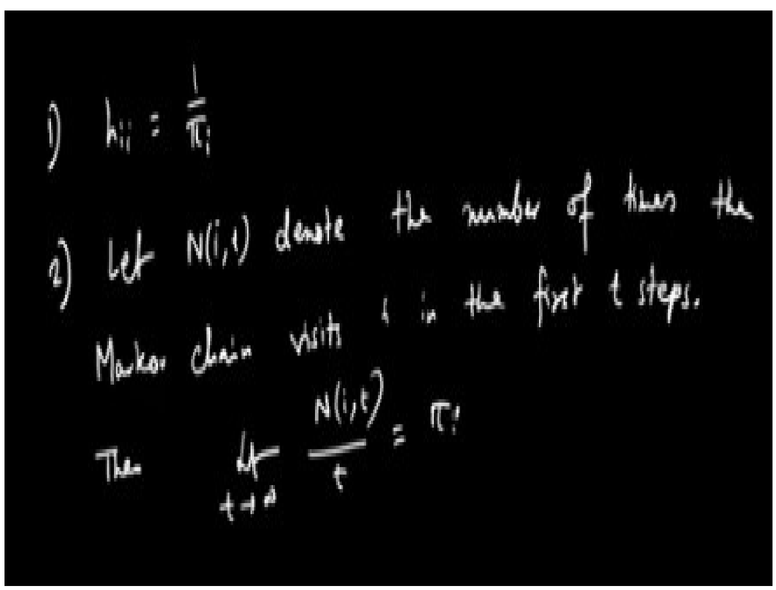

First $h_{ii} = \frac{1}{\pi_i}$ it is like it starts at i and after leaving i what is the expected number of time number of steps it takes to come to i. So it is not 0 it is like it start because in the first step it leaves it can leave and then what is the next earliest j where it again comes back to i; so $h_{ii} = \frac{1}{\pi_i}$. Second let $N(i,t)$ denote thank you the number of times the Markov Chain visits i in the first t steps then $\frac{\lim_{t \to \infty} N(i,t)}{t} = \pi_i$. So we will stop here today we will use this fact to prove some interesting result in the next class thank you.