Selected Topics in Algorithms
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Module No # 05
Lecture No # 24
Radom Walk on Cycles

Welcome so we have been doing the random walk on Markov Chains and in the last class we have seen a random work on the set of all independent sets of a graph. So will continue to see more examples; of random works on graphs.

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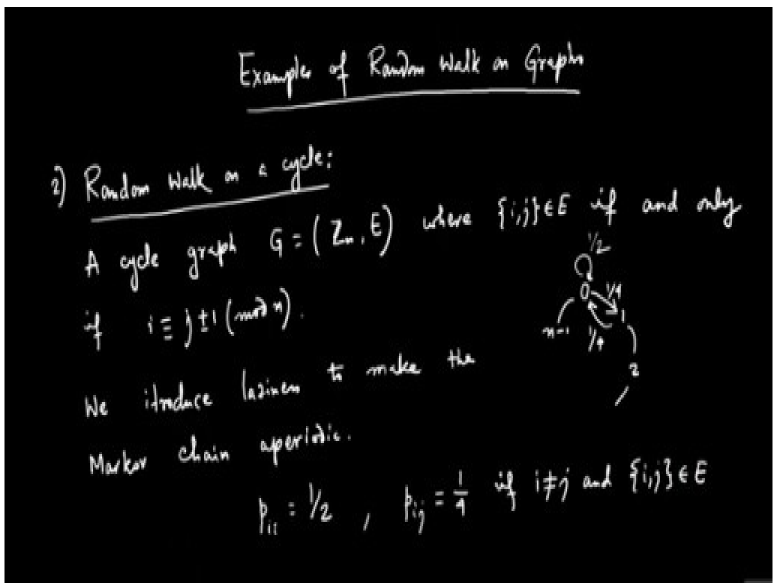

So on first example today is random walk on a cycle, so consider a cycle graph where g equal to vertices are labeled using 0 to n - 1 and there is an edge e if and only if i is $j \pm 1$ congruent n. That means pictorially it looks like this from 0, 1 this is so this is how the graph look like it is a cycle. Now on this graph we will define a Markov Chain and that we do like from each vertex it goes to either side now if n is even then this cycle will be an even cycle and in that case it will not be and even for any n this will be event cycle if n is even.

And then it will not be approved it will be periodic with period 2 so to make it a periodic what we do is that we introduce laziness to make the Markov Chain a periodic means what? That means let us define probabilities of self-loops to be half so from, 0 to 1 it goes with probability to one fourth from 1 to 0 with probability one fourth and this is with probability half. Means p_{ij} is one fourth if $i \neq j$ and there is an edge between i and j.

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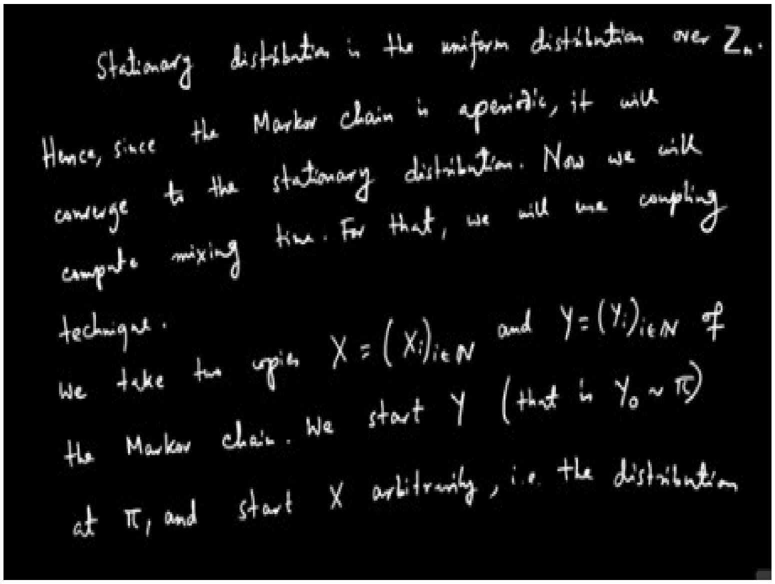

So what is the stationary distribution is the uniform distribution over $\mathbb{Z}_n$. Now what we do is that we so hence since the Markov Chain is aperiodic it will converge to the stationary distribution but we will study how fast are the convergence. Since the Markov Chain is aperiodic it will converge to the stationary distribution. Now we will compute mixing time and for that we will use coupling technique**.**

So what is the idea first we take 2 copies of the Markov Chain so 2 copies let us call it $X = (X_i)_{i \in \mathbb{N}}$ and $Y = (Y_i)_{i \in \mathbb{N}}$ 2 copies of the Markov Chain. The idea is that 1 copy we will start at stationary distribution and the other copy you know is designed in such a way other copy starts at arbitrary distribution. But it is designed in such a way that once they meet in they will be together thereafter.

And because one copy of the, Markov Chain who is always in the stationary distribution so the mixing time will be the expected time to for these 2 Markov Chains to meet that is the idea. So its recall in coupling what is coupling distribution it is a joint distribution such that the corresponding marginal have the right distribution that is what is called coupling. So what we do is that we start one Chain at the stationary distribution.

We start Y that is sample Y_0 using stationary distribution π we start Y at π. Now by the definition of stationary distribution if Y_0 is distributed according to π, Y_1 is also distributed equivalent to π Y_2 is also distributed according to π and so on.

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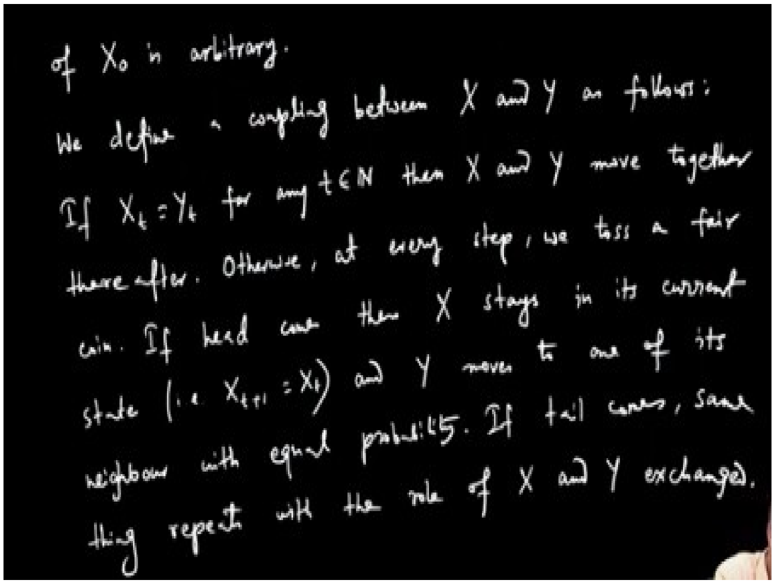

So we start Y at stationary distribution and start X arbitrarily that is the distribution of X_0. Now we define a coupling between X and Y as follows. If $X_t = Y_t$ for any $t \in \mathbb{N}$ then than X and Y move together thereafter**.** What do I mean by that move together that means let one Markov Chain let us say X take the next state according to the transition probabilities and let Y copy that. Because X follows the Markov Chain transition probabilities Y also follows that.

Otherwise at every step expensive $X_t \neq Y_t$ we toss a Fair coin. If head comes then X stays in it is current state and that is $X_{t+1} = X_t$. That means this happens with if it comes which happens with probability half this is the right move for X if X is concerned because you know with probability half it is supposed to stay in its current state. But in this case let Y move so if head comes X will constitute and Y moves to one of its neighbour with equal probability.

That means if head comes then we toss another fair coin and if we again suppose it comes then it moves left and or if it tails come then it moves right. And again you see that Y also follows the correct distribution of Markov Chain it moves to its one of its neighbours with probability one fourth each. If tail come same thing repeats with the role of X and Y exchanged that means if tail comes y stays in its current position and X moves one of its neighbours with equal probability.

Now let us see what we have achieved we will now bound in the expected number of time what is the expected number of times? That X_t is same as Y_t for the first time because once they are same they will be move together thereafter.

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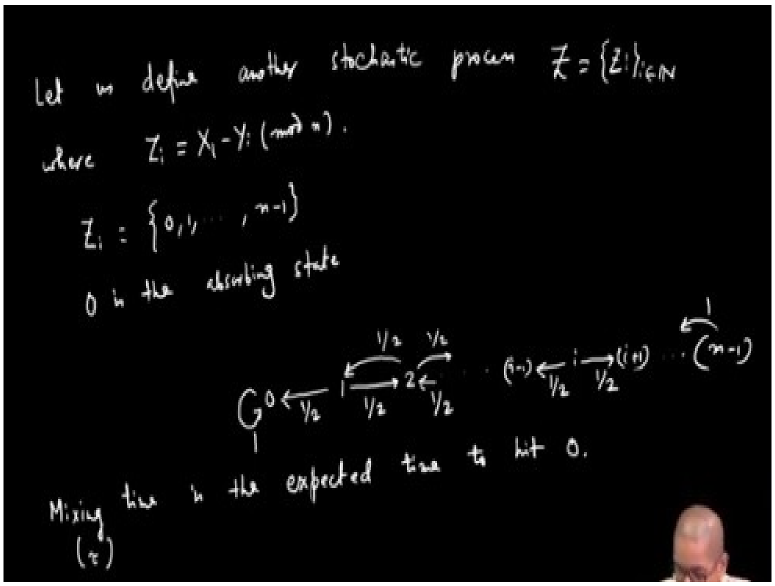

So towards that we define another stochastic process call it Z which is $(Z_i)_{i \in \mathbb{N}}$ where Z_i is defined as $Z_i = X_i - Y_i (mod\, n)$. Now see how does the value of Z change first of all Z_i takes value in this set 0, 1 up to n - 1. And 0 is the absorbing state once Z_i becomes 0 then Z_{i+1} also becomes 0 and so on, so 0 is the absorbing state. So here are the possible values $\{0, 1, 2, n-1\}$ if it is 0 it remains in 0 forever.

And in particular if it is in state i then with probability half Y stays there and X moves to its neighbours so this you know in every step if X and Y are not same then the value of Z changes. Because if a, either if X stays then Y changes and if Y stays exchanges this is when X and Y have not met together. So this is it goes to left with probability half and right with probability half and so this is the how this transition look like is how the transitions for Z process Z look like. And for that we know and when it is n - 1 it will go to n this is 1 it will go to n – 2.

So from so mixing time is the expected time to hit 0 or to reach 0 for Z. And from 2 set analysis let us call mixing time to be τ. So from the analysis of 2 set algorithm the randomized algorithm for 2 set the same behavior was there for 2 set.

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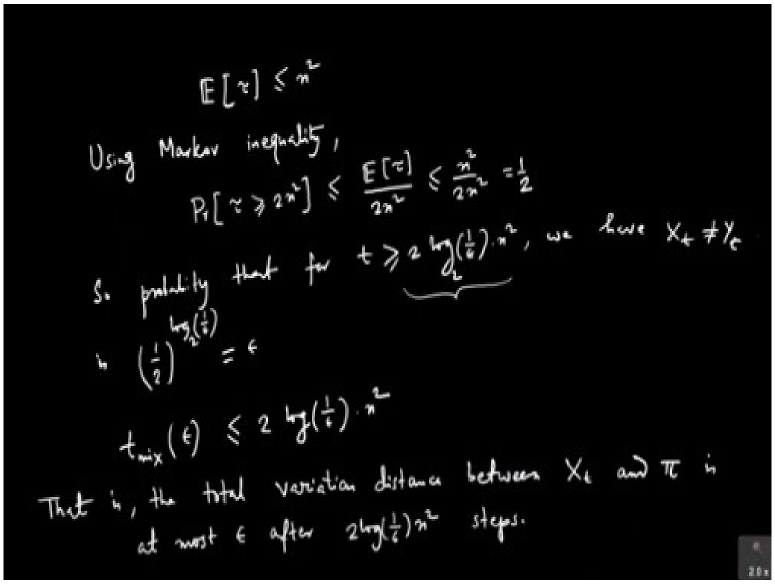

And we observe that expectation of τ is less than equal to n^2 it depends on the start state and even if it starts at n - 1 then it is n^2 if it starts at some state other than n - 1 then this strictly less than n^2. So now using Markov inequality we have probability that tau is greater than equal to twice n, square is less than equal to in, n^2 right. This way $\frac{E[\tau]}{2n^2}$ this is less than equal to $\frac{n^2}{2n^2}$ which is $\frac{1}{2}$.

So probability that tau that for $t \geq 2\log\left(\frac{1}{\epsilon}\right)n^2$ we have $X_t \neq Y_t$. That means these 2 processes have not met even after $2\log\left(\frac{1}{\epsilon}\right)n^2$ in this case is $\left(\frac{1}{2}\right)^{\log_2\left(\frac{1}{\epsilon}\right)}$ which is ϵ. So after $2\log\left(\frac{1}{\epsilon}\right)n^2$ many states the probability that these 2 brothers have not met is less than equal to ϵ.

Hence the mixing time the t_{mix} and if they met the distribution of X_t so this Y_t was always Y, we started Y in π and X was arbitrary. So and if they met at t-th state then the distribution of X_{t+1}

onwards it will be π. So $t_{mix}(\epsilon)$ is less than equal to $t_{mix}(\epsilon)$ is the mixing time when the probability of so t_{mix} so this is from coupling lemma $t_{mix}(\epsilon)$ is less than equal to $2\log\left(\frac{1}{\epsilon}\right)n^2$.

The distance of t of distributions between the distribution of X_t after $t_{mix}(\epsilon)$ many steps and the stationary distribution is less than equal to $2\log\left(\frac{1}{\epsilon}\right)n^2$ many steps. That is let me write that the total variation distance between X_t and π, Y π because this is the distance between X_t and Y_t this is a probability that X_t is not equal to Y_t and total variation distance is at most probability X_t not equal to Y_t.

These from coupling lemma and the distribution of twice Y_t is π so the total variation distance between X_t and π is at most epsilon after $2\log\left(\frac{1}{\epsilon}\right)n^2$steps.

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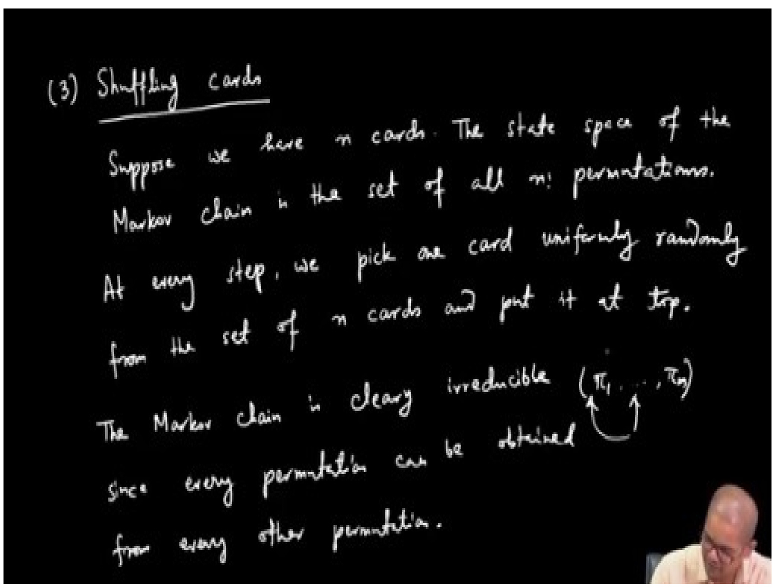

Our next example is shuffling cards so what is the Markov Chain so suppose we have n cards the state space of the Markov Chain is the set of all n factorial permutations of this n cards. And now we will define transition, so the fundamental process of card shuffling is how many times we should shuffle the card so that the all cards are uniformly distributed or saying the same thing in other way. The pack that we have is one of the n factorial permutations uniformly at random.

So what is the process? The process is very simple at every state what is the shuffling process? What is one strip of shuffling at every state we pick one card uniformly randomly from the set of n cards and put it at top. So it is like there are some permutations pi 1 to pi n this is suppose the current deck look like I pick an, card uniform at random and put it bring it at the top and that is how I get the next permutation and so on.

This is how the transitions are defined so again is this Markov Chain irreducible so Markov Chain is clearly reducible because from every permutation by this process you can reach any other permutation. So the Markov Chain is clearly irreducible since every permutation can be obtained from every other permutation. And is it aperiodic of course yes because of self-loops the randomly picked card may be the fast card itself then the permutation does not change.

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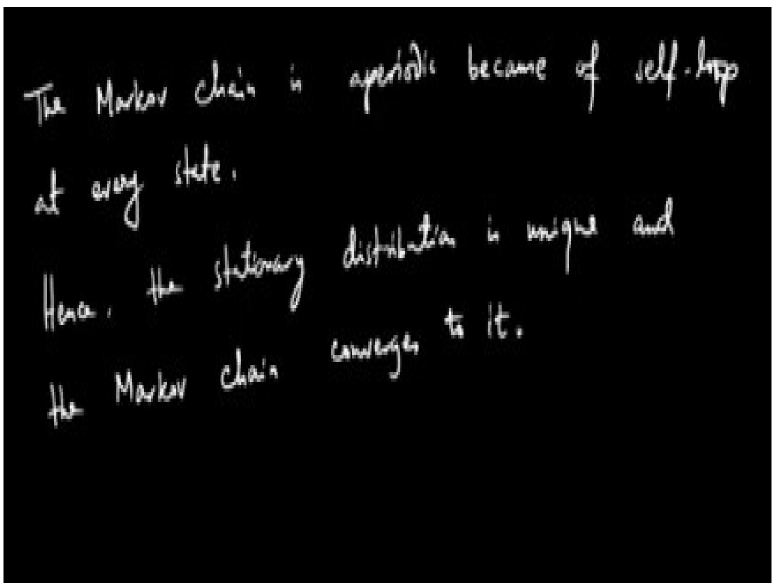

So the Markov Chain is a periodic because of self-Loop at every state. Hence the hence the stationary distribution is unique and the Markov Chain converges to it so we will stop here today.