Selected Topics in Algorithm
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Module No # 05
Lecture No # 23
Metropolis Algorithm, Markov Chain on Independent Sets

Thank you welcome so in the last lecture we started studying Markov Chains and we also have seen the fundamental theorem of Markov Chains and we started studying random walk on graphs and that we will continue into this class.

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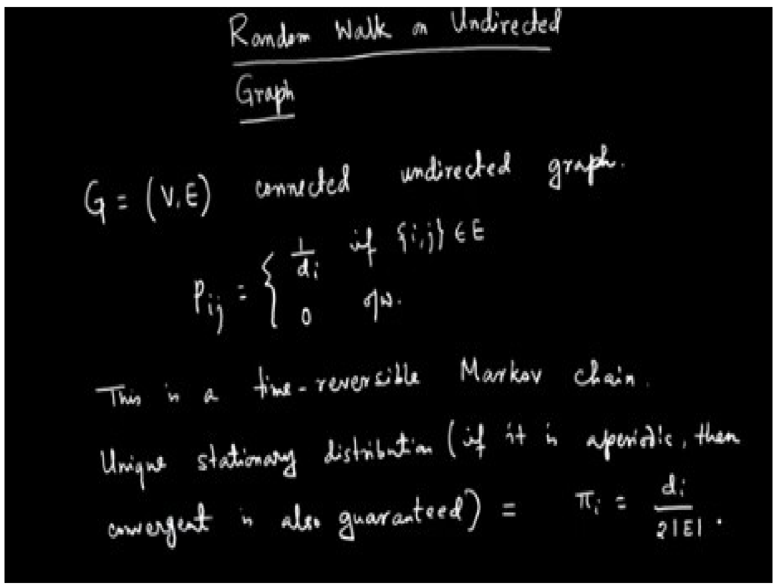

So topic is random walk on undirected graph so let this we have seen in the last class that G suppose $G=(V,E)$ it is a connected undirected graph and this transition probabilities from p_{ij} is 1 by degree of i. If there is an edge between i and j and 0 otherwise and we observe that is a time reversible or it is called time reversible Markov Chain that means it satisfies the detailed balanced equation and the unique stationary distribution.

If it is a periodic unique stationary distribution then it converges also then convergent is also guaranteed is the degrees that π_i is the probabilities the stationary distribution π_i is proportional to degree and so the normalization term is summation degree which is twice the number of edges.

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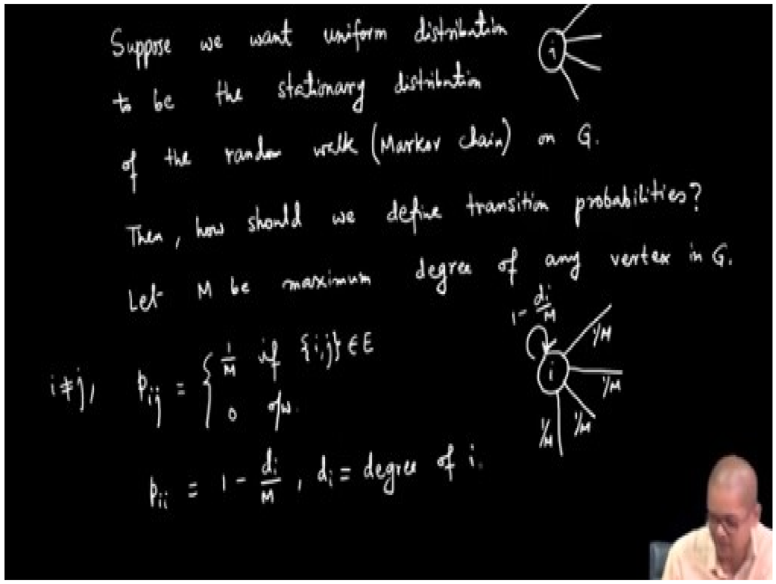

So this is the stationary distribution that we have seen in the last class now suppose if in a graph in an underrated graph if in every iteration on every step of the Markov Chain if the transition happens uniformly over the neighbours. Then this is the stationary distribution but suppose we want stationary say uniform distribution over the state space as stationary distribution then how do we define the transition probabilities let us see that.

Suppose we want uniform distribution to be the stationary distribution of the random work which is a Markov Chain also random walk on G. So suppose I want uniform distribution to be the stationary distribution then how should we define transition probabilities so for that the question is then how should we define transition probabilities. So for that let M be maximum degree of any vertex in G the idea is that defined transition probabilities suppose this is the vertex i any vertex and these probabilities of each edge define it to be $\frac{1}{M}$.

This way the sum of the probabilities of outgoing edges will be degree by M and but this sum of the r transition probabilities of outgoing it should be 1. So to ensure that we add a self loop which with the weight remaining so this is $1-\frac{d_i}{M}$ so now let us define p_{ij}. Let us define formally $i \neq j, p_{ij}=\frac{1}{M}$. If there is an edge between i and j and 0 otherwise and $p_{ii}=1-\frac{d_i}{M}$ where d_i is the degree of i degree of vertex i in G.

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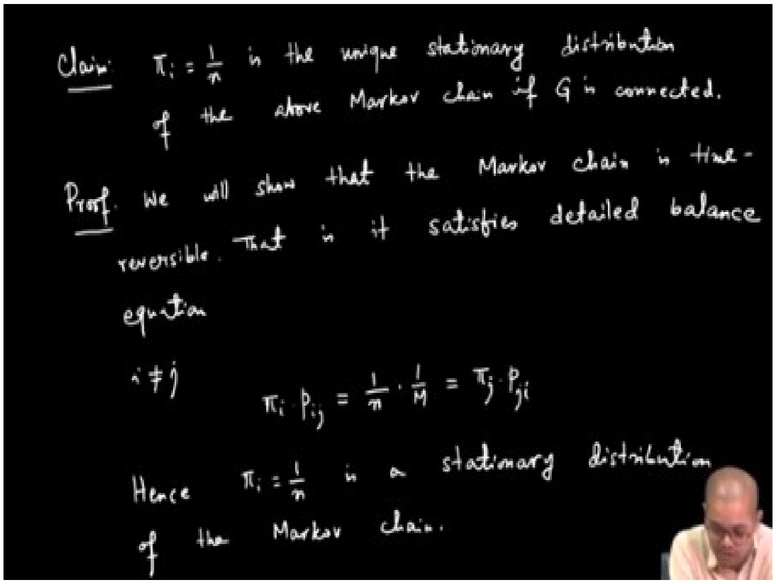

So we claim $\pi_i = \frac{1}{n}$ is the unique stationary distribution of the above Markov Chain if G is connected. So for that again let us define the so for this also we will show that this Markov Chain is time reversible so we will show that the Markov Chain is time reversible that is it satisfies detailed balance equations again. So for that let $i \neq j$ then $p_{ij} = 0$ and also $p_{ji} = 0$ so detailed balance equation is satisfied and otherwise we have $\pi_i p_{ij} = \pi_j p_{ji}$.

So hence $\pi_i = \frac{1}{n}$ is stationary distribution of the Markov Chain and if G is connected then this is the unique stationary distribution it follows from the fundamental theorem of Markov Chain that for irreducible. Markov Chains stationary distribution is unique and on top of it is a periodic then Markov Chain will converge to this distribution.

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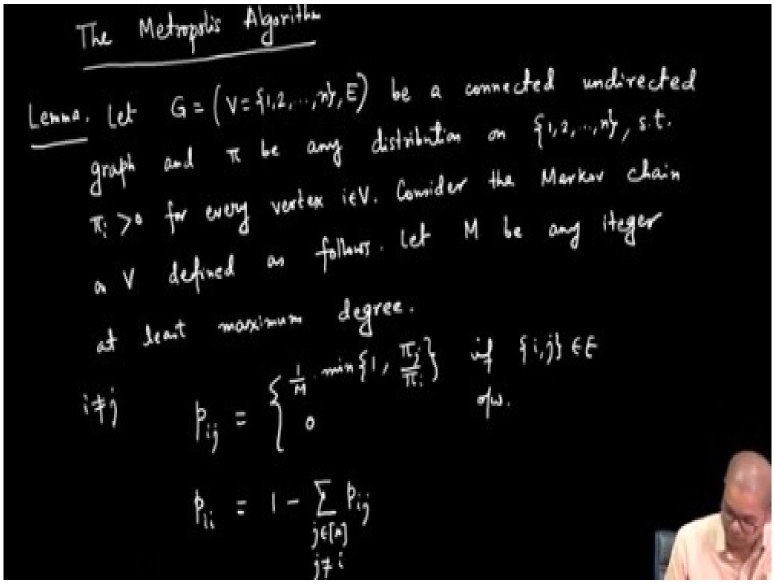

Then you one can ask why only uniform distribution why not any arbitrary distribution given an arbitrary distribution on the set of vertices can, we design Markov Chain with the algorithm is the metropolis algorithm. So what is the acetic so let me write it as a Lemma so let $G[V]=\{1,2,\ldots,n\}$ be a connected undirected graph and π be any distribution on 1 to n such that π_i is greater than 0 for every vertex i V then consider the Markov Chain defined on V defined as follows.

So let as usual M be any integer at least maximum degree define $p_{ij}=\frac{1}{M}$ times minimum of 1 and $\frac{p_{ij}}{\pi_i}$. If there is an edge between i and j and 0 this is if $i\neq j$. Otherwise and p_{ii} is the remaining probability this is $1-\sum p_{ij}$. So this ensures that the sum of the outgoing some of the probabilities of the outgoing edges is 1.

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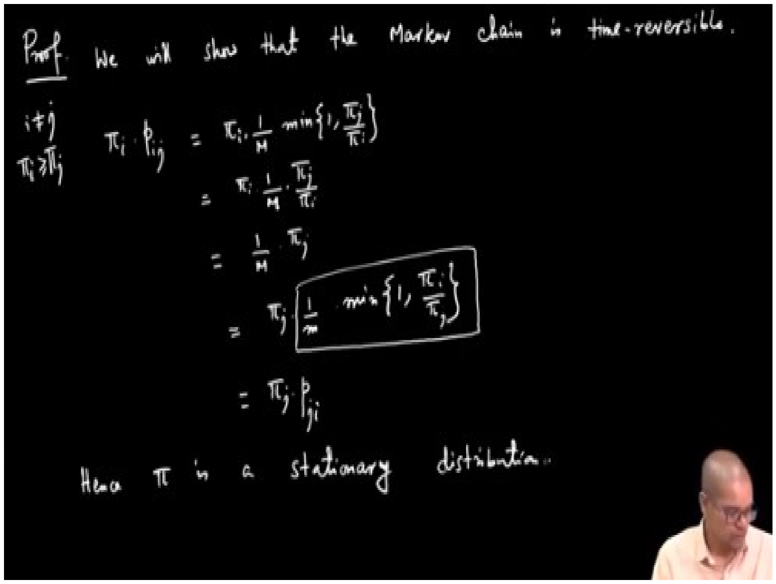

Proof then we will show that the Markov Chain is time reversible that means it satisfies detailed balanced equation. So for that let us say $\pi_i p_{ij}$ is consider any $i \neq j$ and suppose without loss of generality let us assume $\pi_i \geq \pi_j$. Then this is $\pi_i \frac{1}{M} min\{1, \frac{\pi_j}{\pi_i}\}$ but $\pi_i \geq \pi_j$. So this is $\pi_i \frac{1}{M}$ which can be written as $\pi_i \frac{1}{M} \frac{\pi_j}{\pi_i} = \pi_j \frac{1}{M}$.

If $\pi_i \geq \pi_j$ then $\frac{\pi_j}{\pi_i} \leq 1$. So then this is $\frac{1}{M} \pi_j$ which can be written as $\pi_j \frac{1}{M} min\{1, \frac{\pi_i}{\pi_j}\}$ because $\frac{\pi_j}{\pi_i} \leq 1$. This is $\pi_j p_{ji}$ so this π_i is this distribution this given distribution satisfies detailed balance equation hence π is stationary distribution so let us take a few examples.

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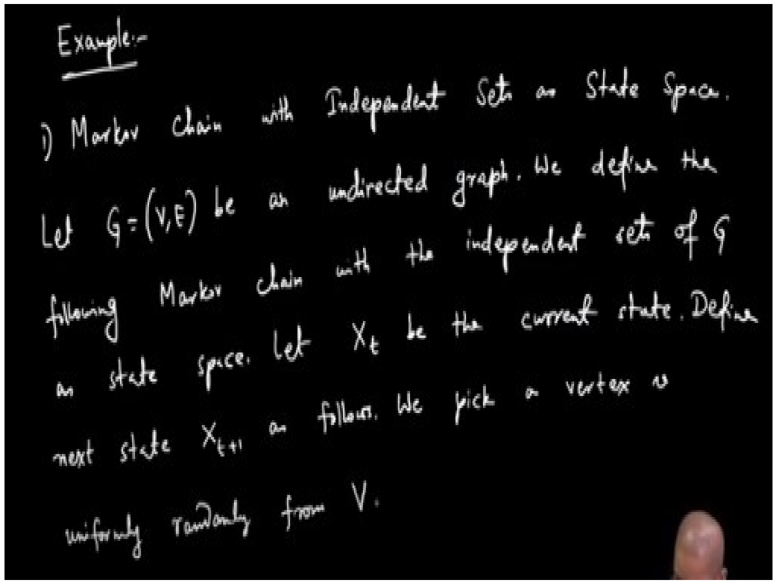

So now we will do some examples of random walks on graphs how we can use it to perform various algorithmic tasks. So our first example is Markov Chain with independent set as state space that means we have a graph G and the set of all independent sets is the state space of a mark option we define a Markov Chain on the set of all independent sets of a graph G. So let $G=(V,E)$ be an undirected graph now we define the following Markov Chain with the independent sets of G, as state space.

So the set of all independent sets of G is at the state space and we need to define the transition probabilities. So let X_t be the current state of the Markov Chain we define next state X_{t+1} as follows X pick so for that we pick a vertex v uniformly randomly from V.

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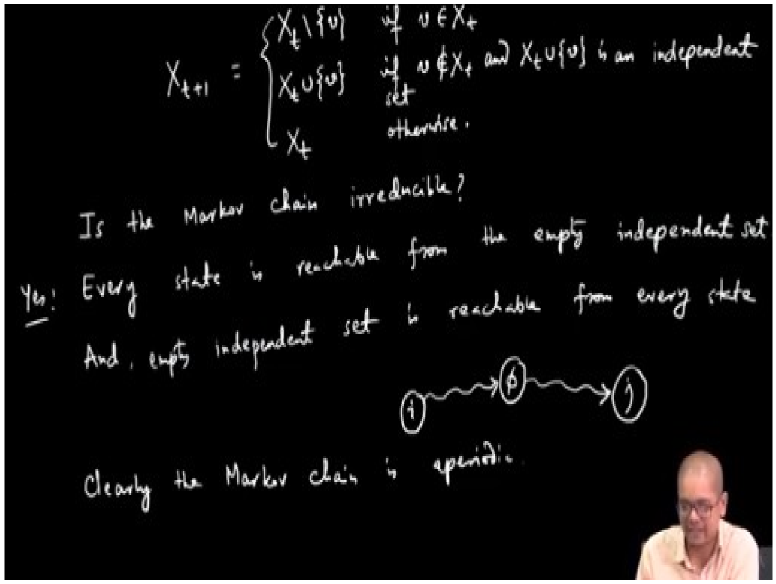

Then X_{t+1} is defined as there are couples of cases if v belongs to X_t that means X_t is an independent set and the uniformly peaked vertex v belongs to X_t. Then the next state which must be an independent set also by v is $X_t - \{v\}$ if X_t is an independent set $X_t - \{v\}$ is also independent j. If v does not belong to X_t and $X_t \cup \{v\}$ is an independent set if this is the case then we define X_{t+1} as $X_t \cup \{v\}$ otherwise we define X_{t+1} as X_t.

Now let us analyze this Markov Chain is this irreducible it means from every step to every step is it connected from every step to every step is there a path. So every state the answer is yes so for that every state is reachable from the empty independent set so the empty set is an independent set empty set of vertices and that it is from that set every state is reachable not only that empty independent set is reachable from every state.

So that means from i to check whether there exists a path or not of course there exists because from i you there is a path to empty set and from empty set there is a path to G. So hence it is irreducible of course it is a periodic clearly the Markov Chain is a periodic since every state except in empty state has a self-loop unless there is and if there is no age in the graph then of course the all set of vertices is also self-loop.

Now all sort of vertices also independency and also that independence does not have a cell flow but except these 2 states all other state has a self-loop.

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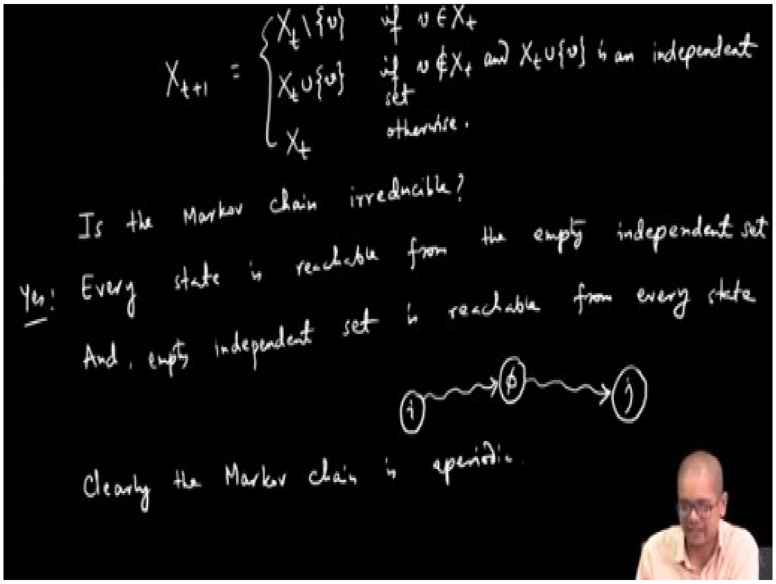

Now hence there exist unique stationary distribution and the Markov Chain converges to it. Now you look at the transition probabilities plate i and j be 2 states i not equal to j then what is p_{ij} so first observe that there exist a transition from i to j if they differ by 1 vertex if they differ by more than 1 vertex then there is no transition. So this is 0 if i symmetric difference j is greater than equal to 2. If they differ these two independent sets differ by at least 2 vertices. If i is symmetric difference j is 1 they differ in exactly one vertex.

Then this is the probability this transition will be taken if the uniformly peak vertex is that vertex which is present in one independence and is not present in the other independent set. So this probability is $\frac{1}{n}$ is the case for i not equal to j. So all the transition probabilities are $\frac{1}{n}$ and we have seen that the stationary distribution in this case is π_i equal to is the uniform distribution over the set of all independent set. So we will stop here today in the next class we will continue seeing some more applications of random walk on graph.