Selected Topics in Algorithms

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Module No # 05
Lecture No # 21
Markov Chain, Periodicity, Stationary Distribution

Welcome so today we will start a new topic which is Markov Chain which is a very important tool for randomized algorithm design.

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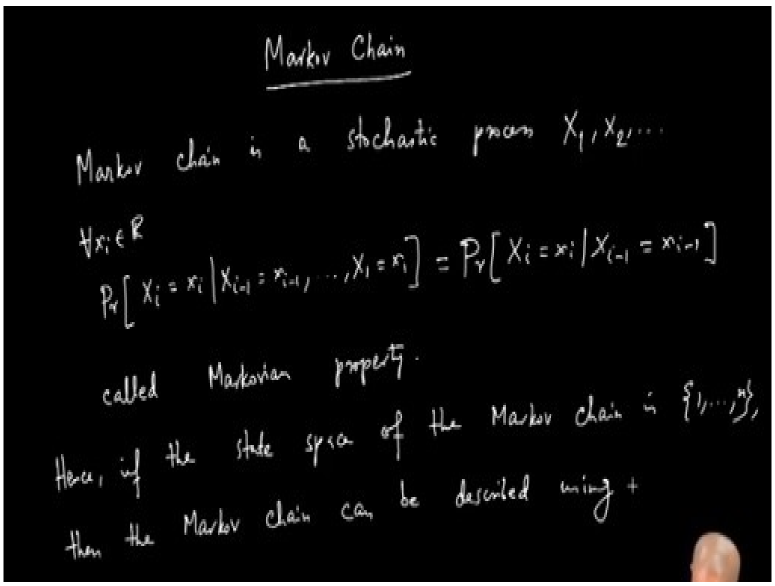

So we now start Markov Chains, so it is a Markov Chain is a, he is a stochastic process; and stochastic process is nothing but a sequence of random variable X_0, X_1 and so on over a probability space. With the property that for all x i in say suppose this random variables take real values then probability that $X_i = x_i$ given. Suppose let us start this index from 1, 2 and so on. $X_i = x_i$ given $X_{i-1} = x_{i-1}, \ldots, X_1 = x_1$.

This is probability of $X_i = x_i$ given entire history is probability of $X_i = x_i$ given the last state this is called the Markovian property. So you know this hence if the state space of the Markov Chain is say 1 to, n.

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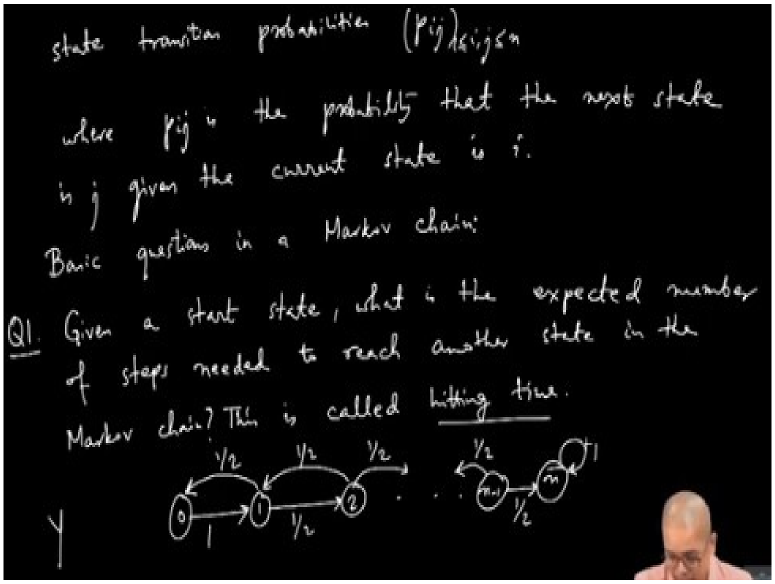

Then the Markov Chain can be described using what is called state transition probabilities p_{ij} s, where i j is from 1 to, n. Where p_{ij} is the probability that the next state is j given the current state is i. So once we have a Markov Chain what are the basic questions that we are interested in so we are often interested in 2 basic questions. So basic questions in a Markov Chain so question 1 given a start state what is the expected number of steps needed to reach another state in the Markov Chain this is called hitting time.

If you recall in the analysis of the or randomized two set algorithm we were interested in hitting time the Markov of state Markov Chain has n + 1 states. Let me write this way say 0, 1 to, n and the transitions are like from 0 it goes to 1 with probability 1 from 1 and this is the mark option y. Let you check that you know that x was not a Markov Chain because that probability depends on history it does not depend only on what is the current state.

So from 1 it goes to again 0 with probability half and 2 with probability half, from 2 it goes to 1 with probability half and it goes to 3 with probability half. From n - 1 it goes to n with probability half and n - 2 with probability half. And in these current absorbing, state once the Markov Chain goes there it stays there forever.

And if you now recall the analysis of randomized 2 set algorithm all we are interested in is that if it start at any state what is the expected number of steps it takes to hit n that is the hitting time.

And we saw that the hitting time of n for the start state 0 is n^2 that is what we showed so this is the question is question one.

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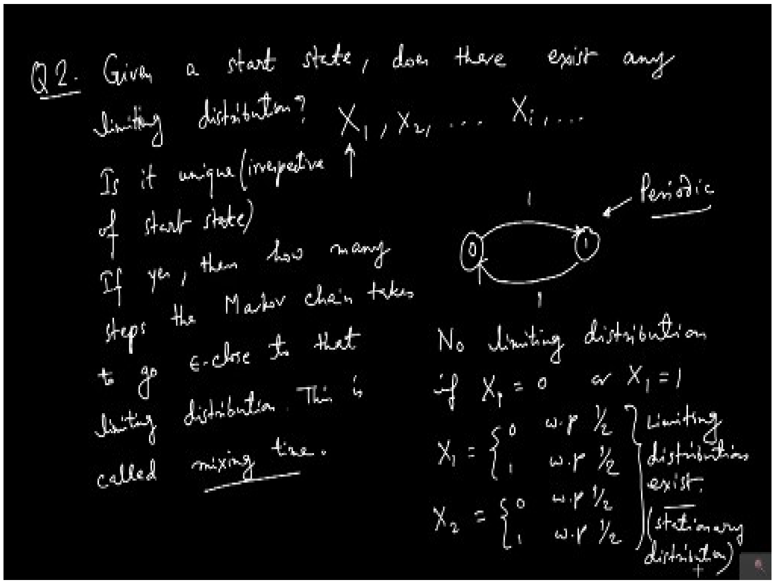

And question two given a start state does there any limiting distribution, does there exist any limiting distribution; that means what this is the; or my Markov Chain $X_1, X_2, \ldots$ and so on. Now suppose $X_1 = x_1$ it starts at some state then what is the distribution of X_i. And this distribution does it does converge so that and if it if it converts so does that converge so that is the first question.

Does that exist any limiting distribution even if it is exist is it unique that means irrespective of start state. So does not matter what is the distribution of X_1 is the limiting distribution unique. So and so for example let us take an example of Markov Chain with 2 states state 0 and state 1. And it has 2 transitions from 0 it goes to 1 with probability 1 and from 1 it goes to 0 with probability 1.

Now you see that you know if this the start state is 0 that means if X_1 is 0, then X_2 is always 1 X_3 is 0 and X_4 is 1 all even states and at even steps the Markov of Chain stays at 1 and at odd steps Markov Chain stays at 0. And you know so hence there is no limiting distribution if the Markov Chain starts at 0 or 1. So in this case it the distribution does not converge no limiting distribution if X_0 is 0 or X_1 is 0 or X_1 is 1 if it starts at 0 or 1.

But if the start state is if X_1 is 0 with probability half and 1 with probability half then the Markov Chain stays there what is the distribution of X_2 then? X_2 will be the probability that X_2 is 0 is the probability that X_1 was 1 which happens with probability half. And the probability that X_2 is 1 is the probability that X_1 is 0 which happens with probability half.

So you see that you know the distribution of X_1, X_2, X_3 all are the same as the distribution of X_1 and in this case limiting distribution is the distribution of X_1 so in this case limiting distribution exist. So suppose so we will see a class of Markov Chains where limiting distribution exists and it is unique. So once we have these two unique limited distribution unit means irrespective of the start state.

Then the next question is so if yes then how many steps in the Markov Chain takes to go epsilon close to that limiting distribution; this is called mixing time this time**.** So we will see all these concepts now. So this sort of Markov Chains is called periodic Markov Chain.

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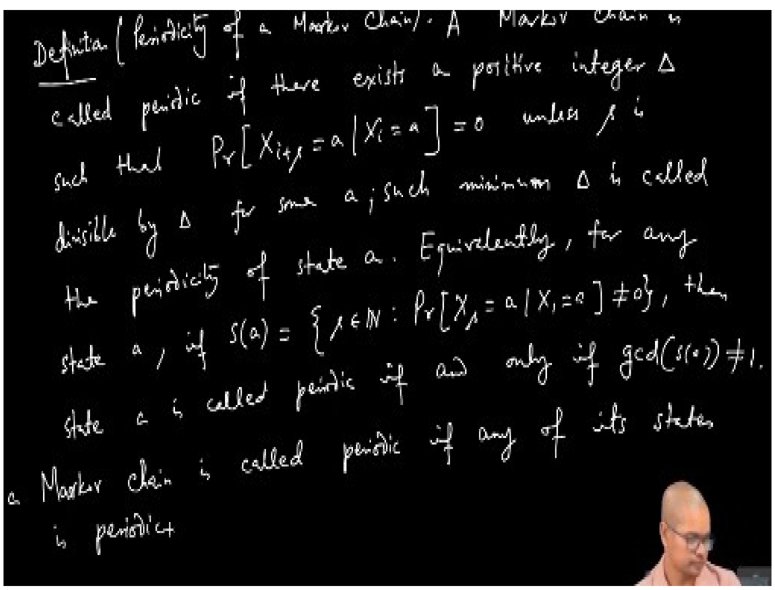

So what is a periodic Markov Chain, let us define definition periodicity of a Markov Chain. A Markov Chain is called periodic if there exists positive integer Δ such that probability that X_{i+s} equal to a given X_i equal to a this is 0 for every a not be s is 0 unless s is divisible by Δ for some a. So a Markov Chain is called periodic if there exists some state a where you know it if at the ith step the Markov Chain is at that state it will be in the after multiples of delta many steps only it will be in that state.

That means probability of X_{i+s} will be 0 given X_i equal to a given X equal to will be 0 unless s is divisible by delta. So such minimum delta is called the periodicity of state a, equivalently for any state a, if $s(a)$, is all natural numbers such that probability X is = a given X_1 = a, is not equal to 0. Equivalently for any state a, this then state a, is called periodic if and only if GCD of $s(a)$, is not equal to 1 and that GCD is called the period. A Markov Chain is called periodic if any of its states is periodic.

So there is a standard technique of converting any periodic Markov Chain to aperiodic Markov Chain. So converting without and worrying by converting that stationary distribution does not change it so that is what.

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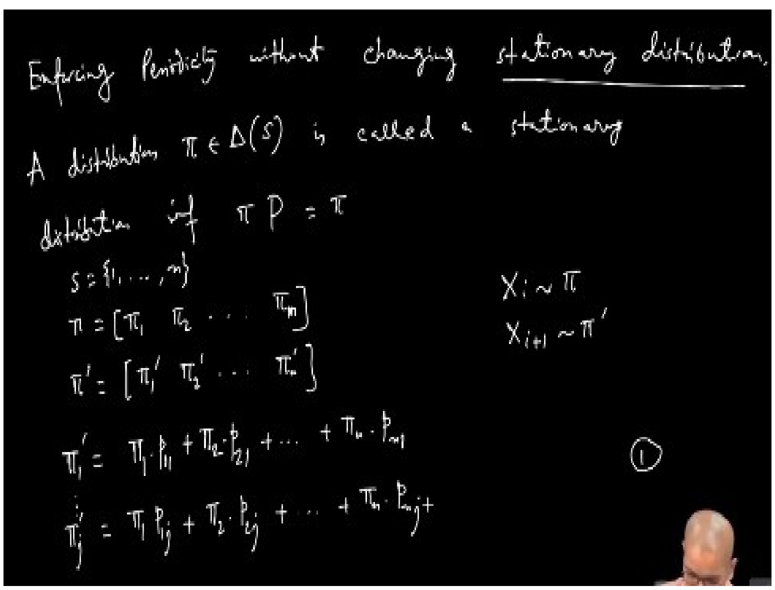

So converting or let me write this way enforcing periodicity without changing what is called stationary distribution. Now what for first let us define what is; stationary distribution? A distribution π over the states is called a stationary distribution. If P is the transition probability matrix $P \times \pi = \pi$. Now what do you mean by $\pi \times P$, let us see π what is π times, π equal to it is a distribution suppose it has n states suppose let s is 1 to, n.

So π is the vector $(\pi_1, \pi_2, \ldots., \pi_n)$ suppose and this is the current distribution of the Markov Chain suppose this is the distribution of X_i. Suppose X_i is distributed according to π that means X_i is 1 with probability π_1 its 2 with probability π_2; and so on let us find out what is the distribution of X_{i+1}. Now X_{i+1} suppose this is π' so what is π'? So π' is suppose it is $\pi'_1, \pi'_2, \ldots, \pi'_n$.

Now how will I compute π'_1? What is the probability that i + 1 is iteration the Markov Chain is in state 1. So π'_1 it is the probability it was in state 1 in the i the iteration times it takes the transition it goes from 1 to 1 + it was in the probability that it is in state 2 times it makes the transition to 1 from 2 it goes to 1. In general $\pi_j \pi'_j = \pi_1 P_1 j + \pi_2 P_{2j} + \ldots + \pi_n P_{nj}$.

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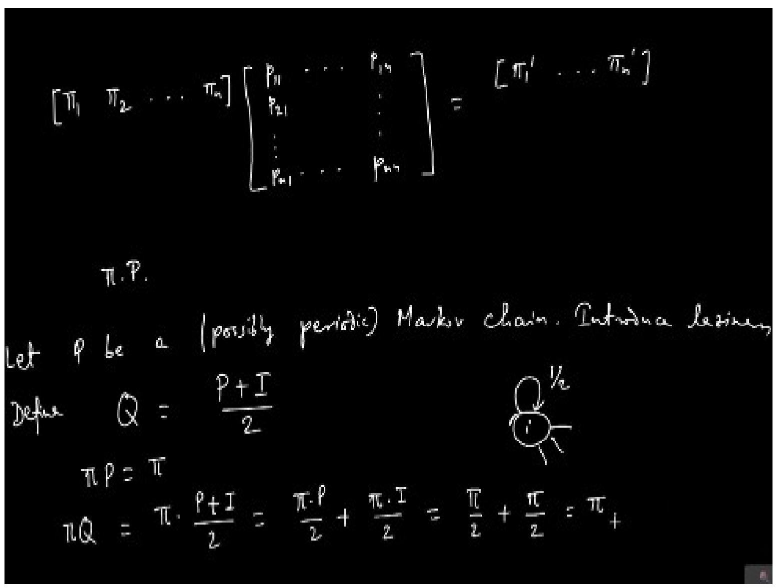

So if the current distribution is pi the next distribution is pi times p this is the distribution of the next state. So now a distribution is called stationary distribution if the distribution of next state is the distribution of the current state. So that means if the Markov Chains distribution is π it is in the stationary distribution then it will remain in the stationary distribution.

So this distribution for this Markov Chain 1 with probability half and 0 width probability half is a stationary distribution of this Markov Chain this is right stationary distribution. Now we will see a technique of converting a Markov Chain or making a Markov Chain aperiodic without hampering the stationary distribution. So let p be a Markov Chain we are possibly periodic, so what we do is that we add self-loop in each state we make it a lazy.

So introduce laziness so at each state i it remains in the state with probability half and with probability half it follows the distribution of p. So define q to be p + what is the distribution of just staying there it is the identity matrix $\frac{I}{2}$. So this is the, consider the Markov Chain with state transition probability q and we claim that stationary distribution does not change. That means if I have a π such that $\pi P=\pi$ then $\pi Q=\pi$.

Let us see what is πQ? $\pi Q=\frac{\pi P+I}{2}=\frac{\pi+\pi}{2}=\pi$. So this way we can make any Markov Chain aperiodic without hampering its stationary distribution so we will stop here today we will continue in the next class.