Selected Topics in Algorithm
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Lecture – 02
Ford – Fulkerson Method

In the last lecture, we have seen the maximum flow problem and with we observed that the natural greedy algorithm does not work.

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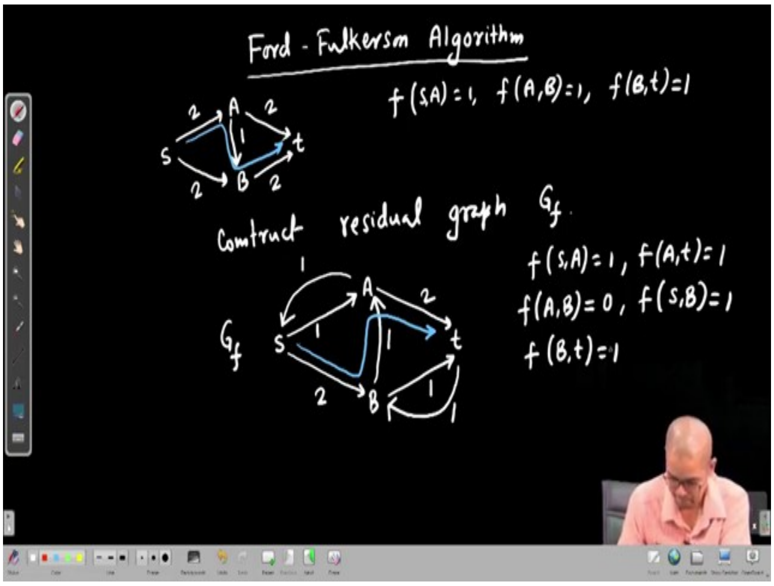

And so, towards that we will fix that problem and get what is called Ford-Fulkerson algorithm. So, what we do is that we construct of a residual flow graph. So, let us go back to that example where the greedy algorithm failed and let us first see the idea in this example. So, the first step remains same. First, you find the PATH, flow path and suppose this is the path and push a flow along this line.

But you know you construct what is called so, our flow f is this is 1 and $f(A,t)$ and $f(S,B)$ is 0. Now, next, when you, when we are trying to find another flow path, another path S to t path to push a flow, do not find the path in the original graph you construct what is called residual graph. So, construct residual graph, let us call G_f. So, what is the graph? s to A to B to t, now, for all original edges we write what is the residual capacity left.

The capacity of S to A is 2 but this A is currently carrying 1 unit of flow, so, the residual capacity is 1. S to B is 2, A to t is 2 and B to t is 1. So, these are the residual capacities but residual graph also contains the reverse edges of the flows. So, there is a flow this edge A to B is carrying a 1 unit of flow and this flow can be reversed. So, in the residual graph I put an edge from B to A of capacity same as the value of the flow in this edge so which is one.

Similarly, S to A carries 1 unit of flow, so, this flow also may be reversed. So, I put an edge from A to S of capacity 1 and here also, I put an edge from t to B of capacity 1. So, here you see that our assumption of the initial graph, not containing any anti parallel edges, becomes more becomes convenient or graph is not cluttered now. So, in the residual graph we can have anti parallel edges.

Of course, so, now, next step is we need to find S to t path in this in in this residual graph G f. This is called G_f because this depends on the flow so, suppose there are suppose it picks S to A to t and it pushes up flow value 1. So, the new flow values are $f(S,A)$ it was 1 the flow in this edge did not change $f(A,t)$ is 1 $f(A,B)$ we write flow values only for the original edges, $f(A,B)$ is 0.

Because it was carrying a flow, A to B 1 unit of flow and now, it is I am pushing 1 unit of flow from B to A they cancels each other. We will formalize it soon but this is intuition. So, f of A, B is 0 and f of S, B is 1 and f of B, t is still 1. So, how does the residual graph look like with respect to this flow?

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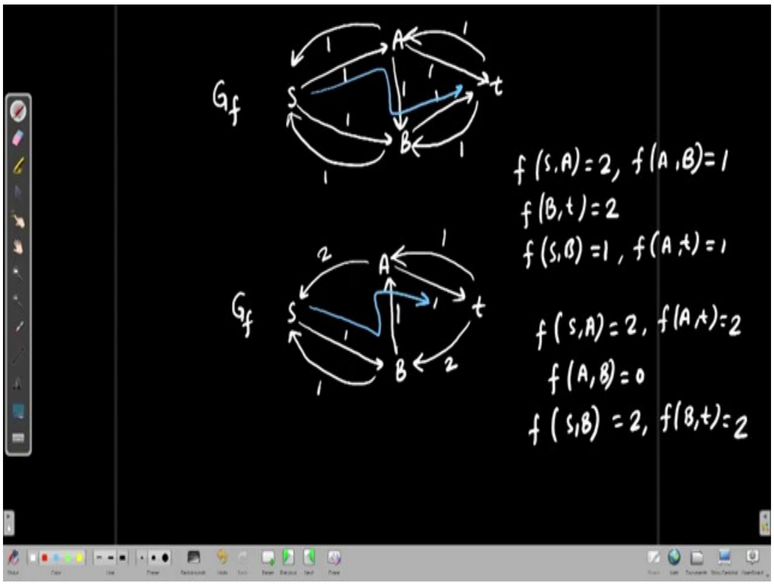

So, in the new G_f S A B t, S to A capacity 1, reverse edge capacity 1, A to t capacity 1, reverse edge capacity 1 A to B capacity 1. This is 1, reverse is also 1. This is 1, reverse is also 1. So, what I do is that again I find the flow path in this particular graph S to t flow path and suppose this is the flow path S to A to t again. Then, how does the capacities flow values will look like $f(S,A)$ is 2, $f(A,B)$ is 1, $f(A,t)$ is 1, $f(S,B)$ is 1 and $f(A,t)$ is 1 $f(B,t)$ will be 2.

So, we can see that again the flow value has increased by 2. The current flow value is 3 in each flow augmentation, flow path augmentation this is called flow augmentation and the flow value increases. And what is the current residual flow? Then G_f , A to B, so, S to A that edge is saturated completely. So only reverse edge is there with capacity 2, S to B carries a flow value of 1. There is a reverse edge for this flow A to B carries the flow value of 1.

So, the corresponding reverse edge B to t there is a flow value of 2, so, this edge is saturated and the reverse edge has capacity 2 and A to t is 1, t to A is 1. Now again, I find A S to t path in this graph. So, let us see and there is only 1 path is 2, A 2 t and if I augment flow along this path then what will be the flow $f(S,A)$ was 2. Now, $f(A,t)$ is also 2, $f(A,B)$ is now 0, $f(S,B)$ is 2, $f(B,t)$ is 2 and the flow value is 4.

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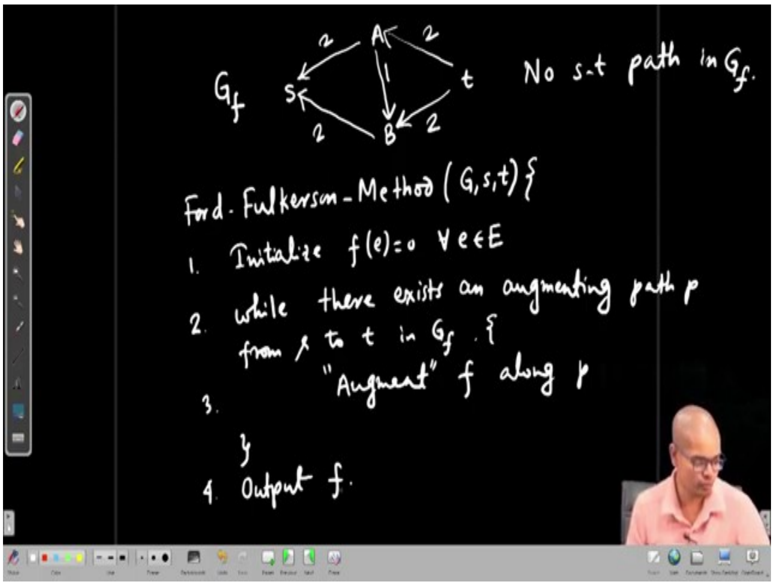

Now let us find the residual graph again. For this flow value S to t, A to B. So, S to A is saturated, so, the reverse edge is there. S to B saturated the reverse edge is there to A to B does not carry any flow. So, this A to t is saturated, so, the reverse edge is there B 2 t is saturated, so, the corresponding reverse edge. Now, I we see that there is no S to t path in G_f .

So, formally, the Ford-Fulkerson method looks like this very simple method, Ford-Fulkerson method takes the graph G, source and sync as input. So, first step is initialize f_e to be 0 for all edge e in e. Second step, while there exists and augmenting path p from S to t in G_f . What we do is that? We do flow augmentation, augment this flow f along p. So, augment means that you see, what is the minimum amount of flow that can be passed along that path?

And you update the flow value so, we will formalize it soon. And when there is no S to t path in G_f then you simply output f. So, what we need to do now is formalize augmenting. What do you mean by augmenting? And formalize residual capacity?

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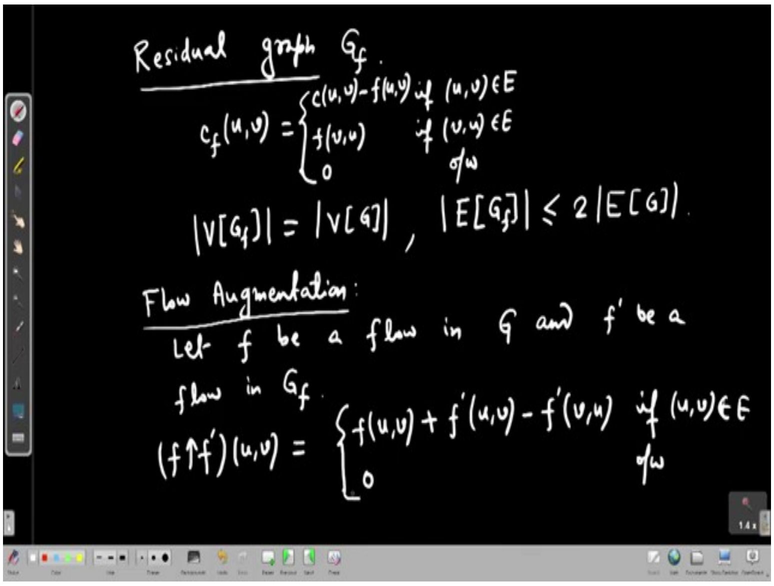

So, let us formally define what is residual graph for a flow G_f ? So, C_f is the capacity of an edge from u to v in the residual graph. So, there are two options, one is if u, v is an edge. There are three options if v u is an edge and the rest is neither u, v nor v is an edge. So, of course, if there is no edge from u to v or v to u then this edge does not exist in the residual graph also, so, this capacity is 0.

Now, if there is an edge from u to v then the residual capacity is simply the capacity remaining in the edge u, v. So, this is $C(u,v) - f(u,v)$ and if the original edge is from v to u then, the in the residual graph I have this is a reverse edge u, v and it is capacity is

$f(v,u)$. So, this is how we define residual graph. You see that the set of vertices in the residual graph remains same from the original graph.

The vertex set remains same but for each edge we may have a reverse edge, so, the edge set may have at most doubled. And now, let us define flow augmentation formally, so, let f be a flow in $G_{f'}$ be a flow in G_f . Then the augmented flow let us we use this notation f' is augmented with f and is a flow in G and so, for each edge u, v all what is the flow value?

The new flow value is the original flow value $f(u,v)$ plus the flows pushed along this edge u, v, f' , u, v minus the flow if any pushed along the reverse edge v, u if u, v is an edge and this is 0 otherwise. So, this flow this is the new augmented flow.

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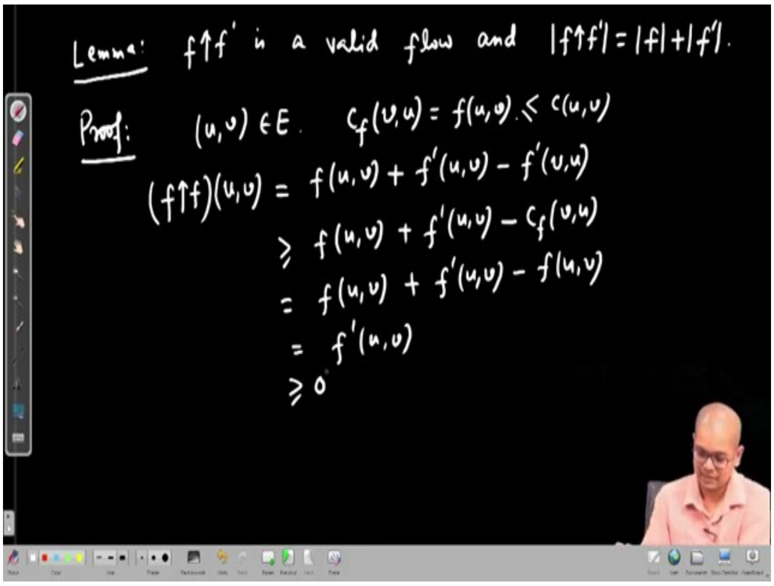

And what we show next that these are these indeed a valid flow. So, here is a lemma that if augmented with f' is valid flow that means it respects the two constraints, capacity, constraint and flow preservation constraint. This and the value of this flowlets which you denote by mod of flow. This is equal to value of f + value of f' proof. So, first let us see capacity constraint.

So, let us pick an edge $(u,v)\in E$ then first observe that what is the capacity of residual reverse edge v in the residual graph $C_f(v,u)$ is nothing but $f(u,v)$. Now, I need to show that after augmentation, the capacity constraint is satisfied. The capacity constraint was originally satisfied. This $f(u,v)$ was less than $C(u,v)$ and we need to show that after augmentation, this value of the flow along this edge u, v, remains less than it is capacity.

So, this is how we defined it this is $f(u,v)+f'(u,v)-f'(v,u)$. Now, this is greater than equal to $f(u,v)+f'(u,v)$ minus the flow along, this reverse edge v, u could be atmosphere it is capacity. So, this is $C_f(v,u)$ so, this is $f(u,v)+f'(u,v)-C_f(v,u)$ is nothing but $f(u,v)$. So, these two cancels and this is $f'(u,v)$.

And this is greater than equal to 0. So, the flow first of all the flow value in this edge u, v is non negative it is not negative.

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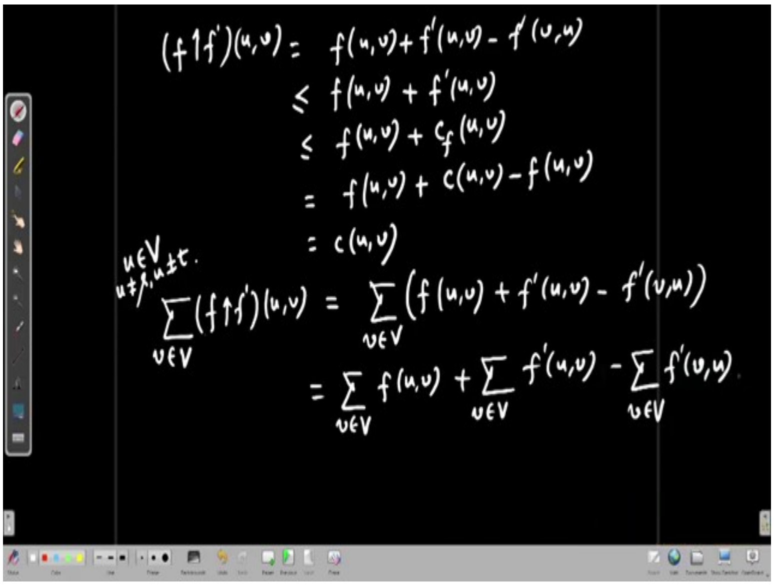

Second, we show that it respects the capacity constraint so, f of f augmented with $f'(u,v)$ is $f(u,v)+f'(u,v)-f'(v,u)$. Now, this is less than equal to I want to show this less than equal to $C(u,v)$. So, $f(u,v)+f'(u,v)$. I can simply drop $f'(v,u)$ term because this is a non-negative term this is a flow in the residual graph. Now, this is less than equal to $f(u,v)+f'(u,v)$ is a flow in the residual graph.

And it should respect the capacity constraint in the residual graph and the capacity of the edge u, v in the residual graph is let us say, $c_f(u,v)$ and so, $f'(u,v)$ is less than equal to $c_f(u,v)$ and $c_f(u,v)$ is nothing but capacity of u, v minus flow of u, v. This is $C(u,v)$. So, after augmenting the flow value along u, v should is less than equal to the capacity of u, v. So, now, let us prove the flow conservation property.

So, the flow conservation property can be equivalent written as for all vertices. So, let us pick, Let us show flow conservation property in a vertex u. And u is not equal to s u is not equal to T. So, if, augmented with f prime u, v this should be 0. Now, this is the flow in u to v flow in this should be equal to flow out. So, let us see what is it? This is $\sum_{v \in V} f(u,v) + f'(u,v) - f'(v,u)$ this is by definition of this flow augmentation.

Now let us push this sum inside. $\sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v) - \sum_{v \in V} f'(v,u)$ Now, if and f' , these are valid flows, so, they respect the flow conservation property.

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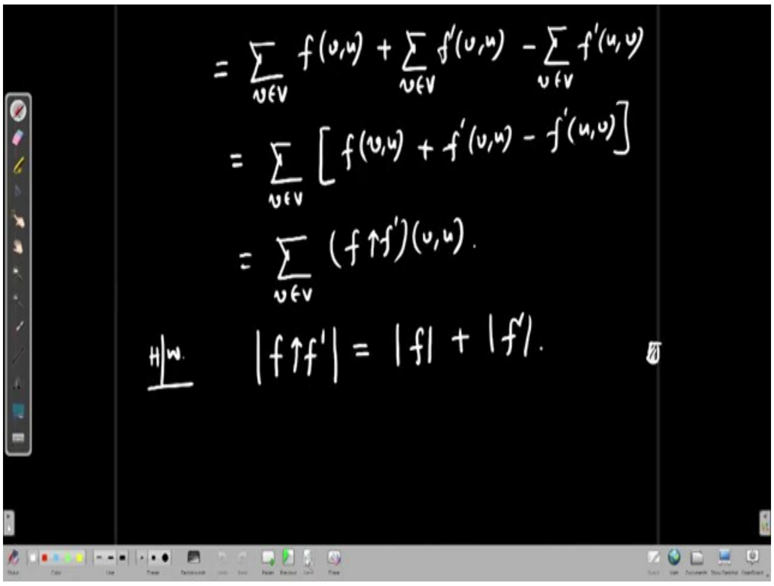

So, this is then for f and f' flow in at v should be equal to flow out of v so, this is $\sum_{v \in V} f(v,u) + \sum_{v \in V} f'(v,u)$ this flow out of v – $\sum_{v \in V} f'(u,v)$. The last term is flow in of v. This is $\sum_{v \in V} f(v,u) + f'(v,u) - f'(u,v)$. But this is nothing but some $v \in V$ these the augmented flow v, so, flow augmentation property, is also satisfied. And similarly, along this line I would let you check give it as a homework.

The last part that the value of the augmented flow is some of the values of the flow which concludes the proof. So, these are valid algorithm let us see is each step can it be done in finite amount of time here, initialize flow this can be done in finite amount of time. And then, while there exists a augmenting path P from S to t in G_f this can be achieved in any by any graph search, algorithm say DFS or BFS or any such search.

You start from S and you see if there is a path from S to t and if there is a path you augment it. The algorithm does not specify which path to space path to pick for augmentation if there is more than 1 path.

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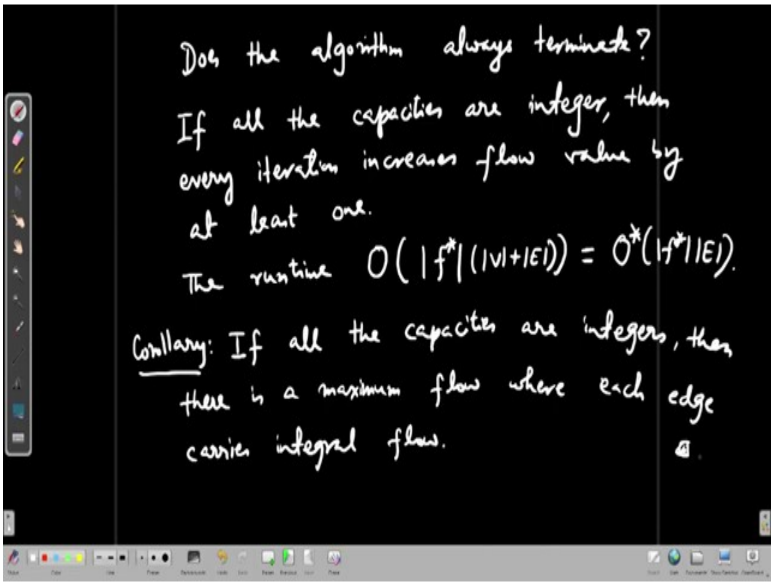

And what about the but does the algorithm terminate? Does the algorithm always terminate? Now, it turns out of course in each iteration the flow value is increasing but it is possible to construct an example with irrational weight values where the algorithm not only loops infinitely but also it does not converge to the optimal value. But if all the capacities are integer then you see that in every step we are increasing then every iteration increases flow value by at least 1.

Hence for integer capacities there exist the runtime of the algorithm in the worst case, it makes at most f star. So, let f star be the mod of f star be the value of the maximum flow. If so many iterations and each iteration can be executed, you in order $V+E$ time by doing a depth first or breadth first search. So, the worst case running time is we go star. Also from this we get an interesting corollary.

If all the capacities are integers then there is a maximum flow where each edge carries integral flow. So, this property will be useful later, as we will see so, we will stop here today.