Foundations of Cyber Physical Systems

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Lecture - 46

Quadratic Program Based Safe Controller Design - Continued

Hello and welcome back to this lecture series on Foundations of Cyber Physical Systems. So continuing from where we left in this section we will be talking about optimal controller design. But also we will try to incorporate safety into the design. Earlier we have seen how to design linear quadratic regulator which is one kind of optimal controller design technique. So fine let us move on.

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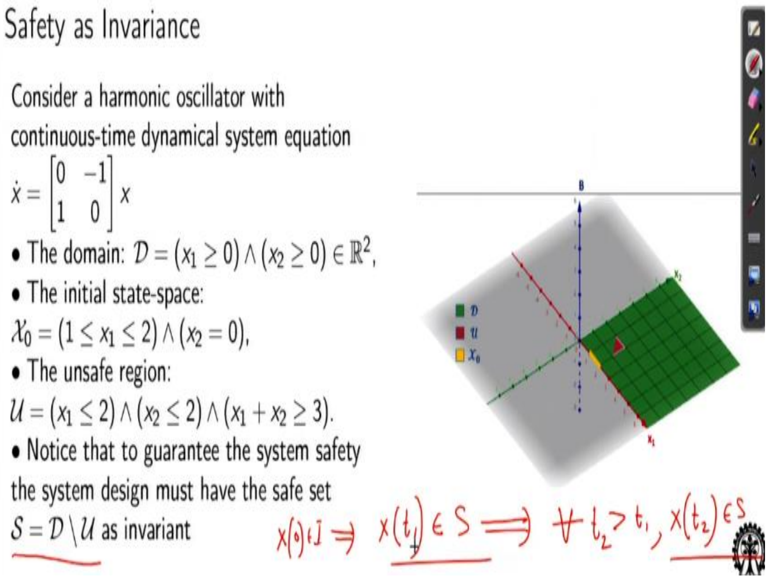

So the first thing we need to do is we need to define safety of the system right. If you recall earlier we talked about barrier functions as a method right, and we will see that how that can be used to ensure safety of the dynamical system that is we will not only design a controller which is optimal but it is also designed in such a way that it ensures that by application of that control law, your system does not wear to some position in the state space which is already marked as unsafe. So we will use this simple example of harmonic oscillator whose this parameters are given like this.

So this is the equation of the harmonic oscillator in a continuous dynamical system form.

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$$

So again we are doing the analysis here in a continuous time space but similarly we can also do it in the discrete space. So and this is the overall domain in the two dimensional space so this is your domain and you can see that this is marked in this space that we want that both the x_1 and x_2 variables to be always being positive.

So in this square in this coordinate two-dimensional coordinate system we are having this green part as the domain of the system okay. And we are saying that well the system may start from an initial state space which is like x_2 equal to zero so you will always be on this x_1 axis right. And x_1 initially is between 1 and 2 right. So that is your initial state space that means the system can start from anywhere on this yellow line here okay. And then we say that well we have an unsafe region in the state space and that unsafe region is given by this set of linear inequalities here.

So let us figure out what are the unsafe regions. So what we are saying is the unsafe region is x_1 less than equal to two. So this is x_1 in this direction less than equal to two. So it is bounded by this line here right and then we have x_2 less than equal to two right .

So x two is increasing along this axis right. So you are bounded by this line here and then we have x_1+x_2 greater than or equal to 3. That means we are saying that well any location on the this side of the line on the right side of this line right so since these three are conjuncted. So overall we have this red mark region as our unsafe regional specification here okay. So what we want to do is we want to design a controller which will steer the system towards stability and additionally it will also ensure that well the system never reaches this thing.

So if you want if you see the difference with our earlier mechanisms earlier we just designed controllers which were optimal with respect to the control objective and the control objective was trying to minimize the deviation from the from the from the equilibrium point and also the control objective was trying to minimize the amount of control effort that is there right. But we are saying that well it does not guarantee that the system will not never be unsafe right.

So how do I also make that control objective sensitive to this kind of a safety specification okay. So we have a safe set here. The safe set is this D minus U, U is my red colored unsafe region.

So this is my safe set. So what we need to ensure is this that the safe set is an invariant now we can understand what it means as an invariant right it means that well let the system at some time t_1 belong to s.

That must imply that for all t_2 that is greater than t_1 the system x at t_2 must belong to S. So this is this is the invariance condition right. And also we have an initial state that means we are assuming that well the initial state is definitely in the safe set and not only that I mean if I start from the initial state eventually I must reach this thing that eventually at some t_1 greater than zero I am I am still in S_1 and then I continue to be in S_1 okay. So this is my requirement now let us see I mean what are the techniques to address this requirement how do I justify this requirement.

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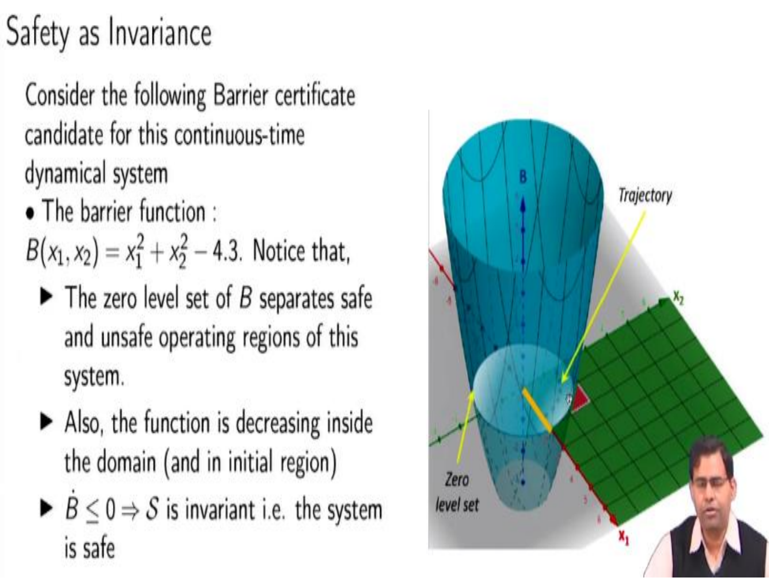

So for this we will use our earlier definitions of barrier function. So we we consider a barrier certificate candidate right we have to if you remember we need to create a barrier function and we have to prove that well this is indeed a barrier function for the system or we have to show that well if something is like a formulation for defining my control law then the control law is such that it always satisfies the barrier function right.

So if you the difference between our earlier and right now method is that well earlier we said that well suppose you are given a system and you were asked to prove that well the system is safe or unsafe. We said that well the system is safe if a barrier function exists for the system. So we took some candidate polynomials and we tried to show that well this is indeed a candidate barrier function or a valid barrier function for this system and since I can define that so the system is kind of safe right. But now the requirement is a bit different we are trying to say that well we will create a candidate barrier function which kind of justifies that well this is indeed a barrier function because it does not contain the unsafe region of the state space and we design a control input so that with that control input the closed loop satisfies the barrier okay. So let us see.

So let us consider our same older barrier function, if you remember which was this x one square plus x two square minus four point three something like that right. So if you if you if you if you try to create the zero level set here it will be circle like this right. It will be a circle like this with radius equal to four point three okay. Now the idea is that well all the points I mean I mean on this on this system.

So this is like a phase plot right because essentially we are plotting the values of $B(x_1,x_2)$ as a function of x_1 and x_2 and the values of b are all defined on this blue shape right on this blue kind of cylinder which is slowly tapering down right. So that is the shape. So the values of b all lie on the surface of this right it can move around like this.

Ignore these black lines here. They actually represent the I mean the way the functions value will go but it will always be on this surface right. Now the thing is this is my zero level set that means if I mean these are the combinations of x_1 and x_2 for which this functions value becomes zero and since we want this to be a barrier function that means the function I mean as long as I am inside the barrier right the idea is that we need to show that well for the system inside its safe region it is always operating with the value of B less than zero.

That means the systems safe trajectories are all located here right on the on the surface of this cone but those positions which are below this zero level set on the surface here right. Now if you remember that why is this a valid barrier function because well we I mean see ideally we could have allowed the system to go here here everywhere but not here right.

We are taking a subset of this allowed safe set and creating a barrier function in that subset and we are saying that well that is my requirement that the idea is that as because I am free to choose the subset as long as nobody is telling me that well where you why your subset is small it could it could be something which also captures this but at the same time that then it will it may not be a continuous convex shape right.

So as long as I am here it is still safe right. As long as I am below it is still safe now the way we design it is that well if we cross the zero level set if we go to some points here like that then what is happening you see maybe this thing is like beyond this boundary the safety boundary here right. But if I cross this barrier it is possible for the systems vary I mean it is possible the system will reach such values of x_1 x_2 so that it comes to the unsafe set. But what will also happen is that suppose sometime I am coming at position here.

If I come at any position here correspondingly the barrier value is positive here right so that is the trick of the barrier function if you remember the moment I come somewhere here it is beyond the zero level set right that for the zero level set the x_1 x_2 2 values are all located here right. The moment I come here the values of the barrier become somewhere here right that means the barrier function becomes positive and that is how we define the barrier function that if.

That means that the moment I cross this the system is like crossing into the unsafe region let us note one thing I may be here it is not unsafe I may be here it is not answered but we have conservatively defined it. What I mean by conservatively defining is let me just repeat we are saying that well this entire green is safe.

The red is unsafe, but let me take a subset of this what we call as safe to be really safe let us which is this part which is basically this part we are considering this part to be really safe. And we are saying that well everything else is unsafe on this side okay. That means now my system can start here and then it can go anywhere here but it is never allowed to get beyond this zero level set because let us understand if I get beyond this zero level set I am also allowing the system to climb to this kind of positions and if I climb to this kind of some position here right then the corresponding x_1 and x_2 values if you take the projection.

Let us say the barrier somehow I allow it to increase if I allow the barrier value to increase beyond zero to a positive value it is possible that I will come here if I come here. That means the barrier value is coming here that means the corresponding x_1 and x_2 values are here right so it becomes unsafe. So we make a subset of the allowed safe region to be my safe set and we make a superset of the allowed unsafe region to be unsafe that means everything here is positive on the top right on the super level sets.

So essentially the zero level set is the safety boundary for me and the sub level sets that means positions which are below this where the barrier is negative value those are my safe regions which is perfect right but I do not allow it to come anywhere here. I do not allow the barrier to grow anything here because if it goes then it becomes possible to for the real system to come here right. So I hope this becomes a bit clearer now. So this was my barriers definition and satisfies the classic barrier definition that it is negative and it is zero at the boundary and it is positive if I ever go to unsafe region so that is how the safety is defined as an invariance right.

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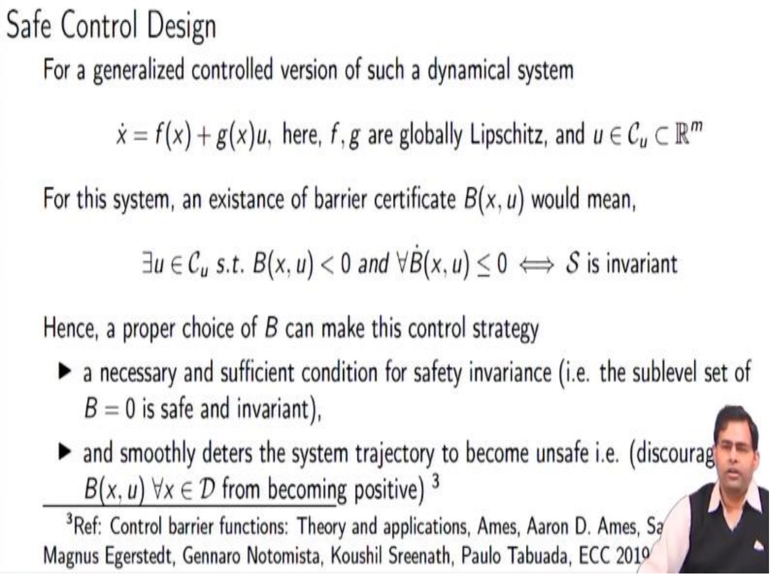

Now what we want to do is we have to design the controller which respects this barrier right so let us cons consider this generalized dynamical system x dot = equal to f (x) plant equation plus g(x) u. So this is the controllers input u and f and g are globally Lipschitz. u is the control input it can be an m array input okay. Now for this for this B function to be a valid barrier certificate now let us notice one thing.

I am trying to define it for the closed loop right that means it is not only defined for x but you also defined over u right. So my requirement is that the control design should be such that the value of u belonging to the set of possible control inputs is such that this barriers value defined over x and u is negative and not only that its gradient is also negative because let us see that that automatically will ensure that my safe set is invariant because if barrier is negative that means by definition of the barrier which was defined with respect to a safe set I am inside this safe region and since the barriers derivative is time derivative is negative that means I can only I can only decrease so I can go here and here but I cannot go here and since I cannot go here, since I cannot come to positions like this on the on the surface of this cone I will never be I mean the values of x_1 x_2 cannot be from this red region. Let me just repeat what I just said.

So we are ensuring by my design of the barrier that initially my barrier value is negative okay. So if you put the value of x_1 x_2 zero you will get minus 4.3. So let us say I am here the barrier is somewhere somewhere negative then you have this initial set. So you can start anywhere inside this you will see that for all values of x_1 x_2 inside this this barriers value is negative. So you will be in the sub label sets okay and then you can climb up here but you are never allowed to go beyond this since you cannot go beyond.

This you cannot come to positions which are like if you take this boundary right if you draw this projection here here like that right, you are unable to come to positions here of the barrier and since the barriers value cannot belong here that means the system cannot come here okay. So that is that is how it works now the thing is we want to design the control so that the barriers value is satisfying these conditions.

So this is how we are going to make a choice of barrier and the control strategy that given the safe region we will try to define the barrier function and we will try to see that well that that this necessary and sufficient condition of safety invariance is satisfied okay. That means the sublevel set of this is because essentially what are we doing. We are we are creating the barrier function in such a way. You are given the safe set let us say right. You are let us say you are given the safe set, what is the target safe set. You choose the safe set now your barrier defining subset as a subset of that okay, and what you do is you ensure that the sub level sets of your barrier defining function is safe and invariant.

That means whatever is given an s as s that you need to be in this s you restrict your trajectory to a smaller part or maybe the full full part of s and you define the barrier function in such a way that this entire s that you have defined is a sub level set or up to the zero set. So if you remember zero set means where the barrier function value is zero the sub level means the barrier function value is negative, the super level means the barrier function value is for the function value is positive. That is how these sets are defined.

So what we just did let me just repeat. This is my entire allowed region the green region. I said that okay I will let my system to only belong inside this region not even in this green region and how am I doing it is I am defining a barrier function so that this my target safety region is contained inside this zero level set and the sub level sets okay. That means whenever my system is moving around in these points what is happening what is happening is that the corresponding barrier values are either located on this surface the zero set or they are located inside this that means in inside any of these sub level sets.

So these are the sub level sets right. That means collection of points on the on the surface but all those values are negative the B values are all negative down downstream right. But the next thing we need to ensure is we should not forget the control. The control cannot be abrupt right the systems control should ensure that the system is smooth. It smoothly steers the systems movements trajectory and also prevents it from becoming unsafe unsafe right. So that means it will discourage the value of B(x,u) from becoming positive and also if you remember our earlier ideas of smooth control design that the control input should not vary abruptly. That also we need to see okay.

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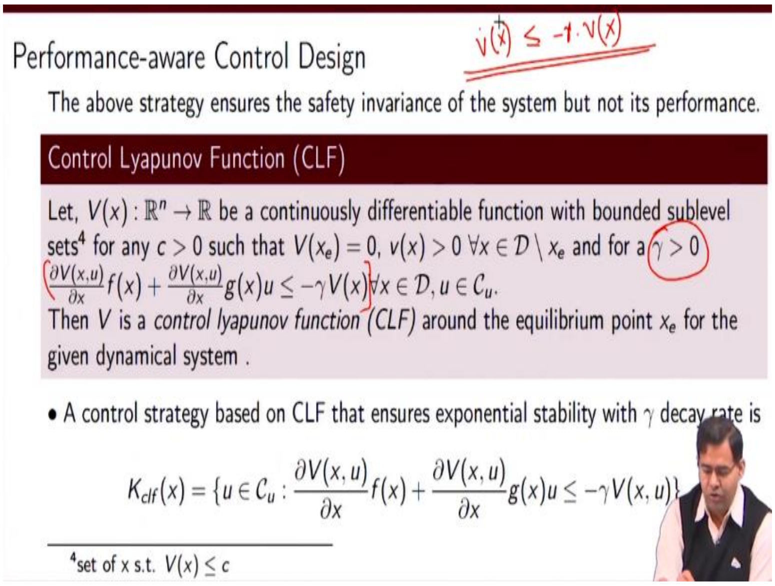

So with these things in mind what we will do is we will first extend our definition of the Lyapunov function. If you remember what the Lyapunov function was doing it was ensuring stability we will define something called a control Lyapunov function which is doing the same thing but for the control system that means the closed loop including the controller. So it is basically the same thing. Why are we we be be be coming with this thing? Because not only the designing the control needs to ensure safety as an invariance of the system but also we have to we have to ensure its performance smoothness etcetera and stability right.

So for this we bring in the control Lyapunov function. It is almost similar as the normal Lyapunov function as you can see that v is a function like this it is continuously differentiable and it has bounded sublevel sets right. That means what that what is what I mean for any value of c. So if you remember that V(x) equal to c. For all values of x for which V(x) equal to are its

I mean level sets for x equal to c. Sublevel sets means sublevel sets for any value c means all values for which V(x) less than or equal to c and we are saying that this is continuously differentiable and it has bounded sublevel sets that means all these sublevel sets are invariant for the function. That means let the system start with some valuation v which is less than some value c. The design of the function must be such that it will we it will always be inside it will always be satisfying this V(x) equal to c that means every sublevel set would be invariant. Now how do we really ensure that?

We ensure that in a way which is very standard in Lyapunov function definition. We ensure that at the equilibrium point the value is zero, at x_e the equilibrium point or $V(x_e)$ equal to zero assuming zero is equilibrium point. In general V is always positive and for all the other values of the positions apart from the equilibrium point we have this condition. The condition is this that this del V del x times f x plus del V del x times g(x) u is less than or equal to some gamma times V(x). Now see this is a bit tricky right. Why do I certainly have something like this if you remember our conditions of the original Lyapunov function, they were like V(x) is greater than zero V dot x was less than equal to zero right and V(0) was equal to zero right.

So it was something like that. Essentially we have the same thing here the difference is see now we have V(x) greater than zero V at the equilibrium equal to zero. And if you remember we need the time derivative to be negative right. But now we are talking about the controlled system. So instead of having a having x dot equal to f(x) we have f(x) plus g(x) u that means the control input is also there right. So when we take the derivative of the Lyapunov function right you take d V dt what you have is del V del x times del x del t right. Now that would give you this del V del x. If you take f dot x dot now x dot is not only f x right.

So you have f(x) plus you have del V del x times this other part which is g(x) times u. So that is what we are writing the original definition of Lyapunov function had this thing as negative right. Just we want that in a different way. So what we are saying is not only is this negative, it cannot be it is not only any negative value but let me give a bound that how much negative it is at different positions of the space okay. So what it really means let us understand. So to give it to have some control on this Lyapunov functions value we write minus gamma right times the value of V(x). So if I now just write it in a simple way.

Essentially we are saying this entire thing if you write this equation. This entire thing it is saying nothing but the time derivative of x is less than or equal to minus gamma times V(x). This equation looks something like an exponential decay right. So that means we are saying that well the Lyapunov functions value must decrease and this value of gamma, gamma being a positive value, is a parameter which is helping the designer to choose in what trajectory of exponential decay the Lyapunov functions value will decrease. Let us understand this again if you just say V dot x is less less than zero.

That means you are just ensuring that well it is negative. But I want to have some control over the performance of the system. So I want to say that well it will not only decrease it will decrease at some rate that I choose. So I say that let it decrease exponentially with a rate that is gamma. Now to ensure that I actually give the exponential decay parameter and I make the parameter as minus gamma times V(x). Because if this is so. Now if you solve this inequality you will see that the value of V will be decreasing exponentially with this gamma as the rate of exponential decay okay.

So essentially what we are doing is we are not only now choosing the Lyapunov function, we are choosing the Lyapunov function as a function of both x and u and we are going to say that well let the control design method give me a value of u so that this inequality is satisfied. Ultimately what is the control design method it is an optimization problem and all we are doing is we are adding constraints on the optimization problem so that certain performance conditions are satisfied.

So here the performance condition that is satisfied is we are forcing the u to be such that this equation is satisfied if this equation is satisfied V will decrease along the trajectory of the system following an exponential decay which in effect will give me a control on how smoothly how fast the system approaches its equilibrium point right. So that is the idea here so this v that we have defined here as function of both x and u along with this parameter gamma for smoothness is known as a control Lyapunov function okay.

So if we if we clear a control strategy now my control strategy can be defined like this it is defined based on this control Lyapunov function that way let me choose a u, so that this thing is satisfied. If this thing is satisfied for a given u then I am ensuring that smoothness factor is there so fine let us let us proceed so this is about performance and now we have already earlier talked about safety. Now all we will do is we will punch these two things together.

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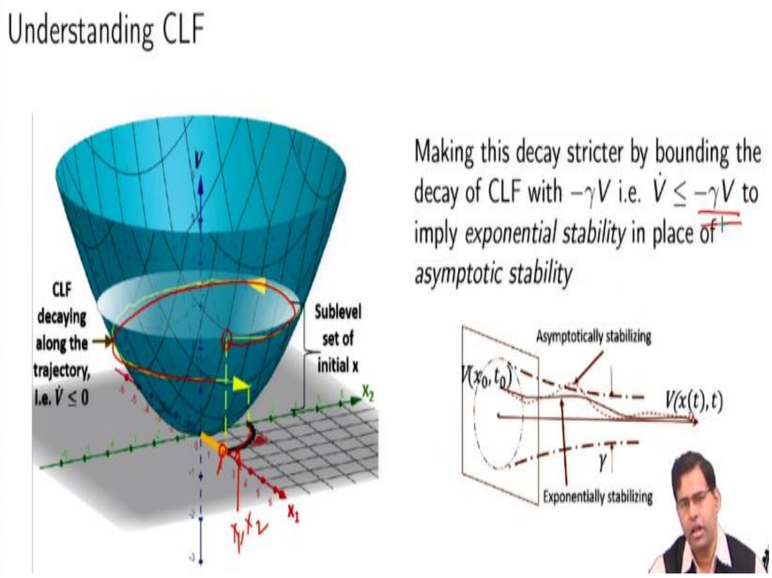

But before doing that lets just have a pictorial view of this Lyapunov function. So like I said what we are doing what we are assuring is well this is my Lyapunov function right and this is the picture of a Lyapunov function. That we want it to be something like this why like this it is always positive right I mean again I am allowing the system to wear on the surface of this this blue shape like an like an inverted igloo let us say or it is it is like a it is like a cup here. What we are ensuring is that the systems trajectory is always on the surface okay.

Now if I if you see this white circle it is a sublevel set of some initial value x okay. That means from here all the values will go to be below that value of x okay and the CLF will decay along the trajectory because the derivative is negative okay. Now if it is negative then well I will always come down like this I and I will be moving around on the surface of the system okay. So let us see the difference that how having the gamma parameter helped me here. So it helps me to make this decay of the Lyapunov function stricter.

Like I said that it is not only negative it is not only decaying it is decaying at a given rate and this is this is what we call as I mean we are ensuring exponential stability with the rate of decay given by gamma rate of exponential decay given by gamma. Instead of just giving this which was for asymptotic stability. So in picture if you see that this is what is this is the this is the exponential decay envelope with the parameter gamma.

So this is ensuring that the system is decaying with within the within this envelope I i have exponential stability here and had it been asymptotic stability it could have gone out like this right. But still well eventually it is decreasing but it can if it may happen that it is not decreasing at this rate exponential of the exponential decay that I wanted okay. So that is the difference and this is how with control Lyapunov function not only I have this shape of the Lyapunov function but also I am able to ensure that well I decay at a rate that I choose and the rate is exponential okay. So that is that is about the stability and performance.

I am I am able to ensure by using a suitable control Lyapunov function that how smoothly the system will approach its equilibrium point okay. But still if I just use this thing there is no guarantee that I do not ion I will be safe because as you can see that well if I just use this well my system has no guarantee that it can start anywhere in the initial point right for which the control Lyapunov functions value. Let us say let us say I am starting here because this is a valid initial point right. So here my control Lyapunov function's value is this now let us say I am moving inside like this so essentially what is happening is the control Lyapunov functions value is kind of decreasing right. As you can see that as my system.

So this is where I am plotting the x_1, x_2 as my x_1 x_2 is changing I am moving like this let us say my control Lyapunov functions value is moving like this of course it is positive but it is decreasing and if I plot it here the value of V as a function of time if I plot it you see that well it is decreasing like this and somehow we are able to also guarantee that it is it is satisfying this rate of decay because with that only I am enforcing the constraint fine. So fine we understand how to give the Lyapunov function a parameter and how to make the Lyapunov function sensitive to both x and u.

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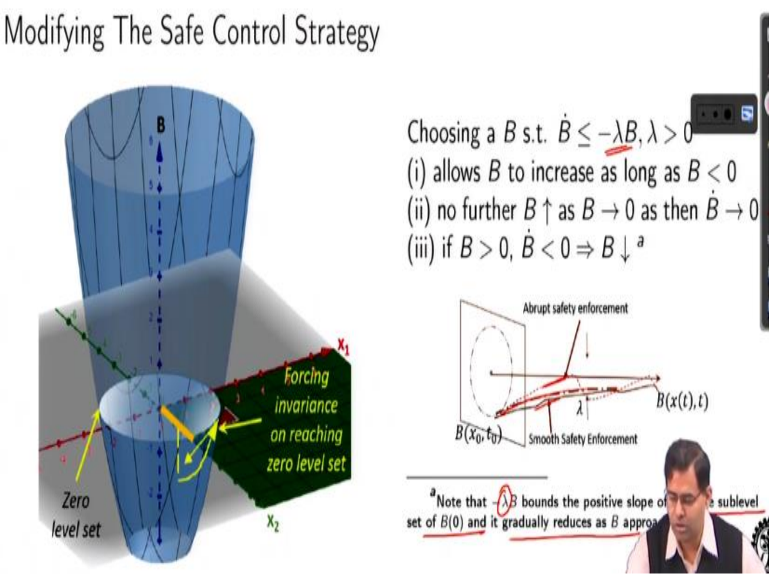

So now how about this safe control strategy so let us understand one thing for Lyapunov function all the subsets all the all the all these sublevel sets they are all invariant. That means if I make my system if I make my system start here at some V(x) less than c it will always be there. Any subset is invariant. But for the for the safety requirement I do not need that for the barrier function all the sublevel sets of the barrier function are invariance that okay.

I only need the barrier function the the the zero level set which is ensuring the safety to be invariant. Let us understand what I mean when I am having this Lyapunov function suppose I choose any set okay for sublevel set of some value x less than c. Suppose this is this height is c. c_1 let us say. So this is my level set for this is this entire thing below this red circle is my sublevel set for c_1 similarly let us say this is c_2.

This circle any everything below this circle is the sublevel set for c_2. So the Lyapunov function guarantees that well if you start somewhere below c_1 you will always be below c_1. If you start somewhere below c_2 you will always be below c_2 I do not want those things to happen for the barrier function. Those are if they happen they are fine. But that is not my requirement my only requirement for the barrier function is that well when I am satisfying the safety I should always be safe that means what that means this is my zero level set all the I mean the I mean the sublevel set of this zero level set should be an invariant.

I should never go here and if I start anywhere here, if I start anywhere inside this, I should be always inside.

This that is all I want because then it automatically guarantees that I never come here. So let us see that how we come with that requirement. So what we do is we choose a barrier value so that this thing is satisfied. Now let us see what does this mean.

So we are saying that lambda is a positive quantity and B dot is less than or equal to some lambda B okay. Now now there is an issue here you need to understand. See initially my B is negative right, I allow my B to be negative inside the safe set. Now if this is negative and I have lambda which is positive and here I have a negative sign that means B dot is allowed to be positive. Is that okay? Why this is okay let us understand. Like I said that I only want my system to be inside this only this should be my invariant region right I want to guarantee.

That means I am fine with allowing the systems value the system's the barrier functions value to increase as long as I do not cross the boundary right. It is not that I always want the barrier functions value to decrease that is my point. If it decreases that is fine it is negative so it is as long as B dot is less than some positive value c it can be positive up to c but also it can be negative right. So it can decrease. But what I am saying is only this is the invariant set. That means I do not need to need B to always decrease. It can sometimes increase but the only thing I need is it should not increase when it comes here. It should not increase beyond the zero level set that is what I want. How do I ensure that? Very simple.

We use this kind of a constraint. It is it is helping me you see. As long as B is negative the value of barrier itself is negative right. As long as the value of barrier itself is negative that means I am somewhere here I am saying that the derivative is less than a positive value. That means the derivative can be negative as well as positive, that means I can be anywhere here I can increase but when can when shall I stop increasing. So the moment I reach to B equal to zero this B dot becomes a force to be less than equal to zero okay.

So in summary I can say this that I am allowing B to increase as long as B is negative but as B approaches zero B dot also approaches zero. That means once I am approaching this boundary I am not allowing B to increase and once if somehow I cross this boundary again this derivative will be negative so B will decrease right. So if if I choose so if you if you recall our earlier earlier barrier function definition we said that well let B dot be always negative.

Just like we introduced the parameter gamma for the control Lyapunov function for the control barrier function I want control right. I want to control that how smoothly the system approaches the safety boundary. So similarly I introduced this parameter. It is ensuring that well I can I can grow but I can grow only up to the safety boundary. I will not cross it the moment I cross it the barrier function's sign will change immediately the gradient will become negative immediately I have to force get back to the system right.

So in this way I am constrained to be inside the safe set and only this zero level I mean sublevel sets of zero level set of zero is my invariant that is what we are able to constrain here okay. So suppose I did not have this thing I just had B dot less than zero. What would have happened, it would have happened that my safety inverse safety enforcement would have been abrupt that means I may have gone like this and immediately I came down right.

But now I introduced this smoothness parameter lambda that means with this I am how the barrier functions value is changing it is not abrupt it is not abrupt like this I have a control here right. So as I approach the safety boundary my barrier functions value will change smoothly like this okay. So that is my point. I mean I am able to force the smooth as a smoothness around this system and it is it is not changing in this kind of abrupt way okay. So using this I am bounding the positive slope inside this sub level set of B zero and it will gradually reduce the value of B as it approaches zero there will be no sudden change. Since there is no sudden change the control will be smooth and the and the guidance of the system would also be smooth in that way. So fine let us proceed from here.

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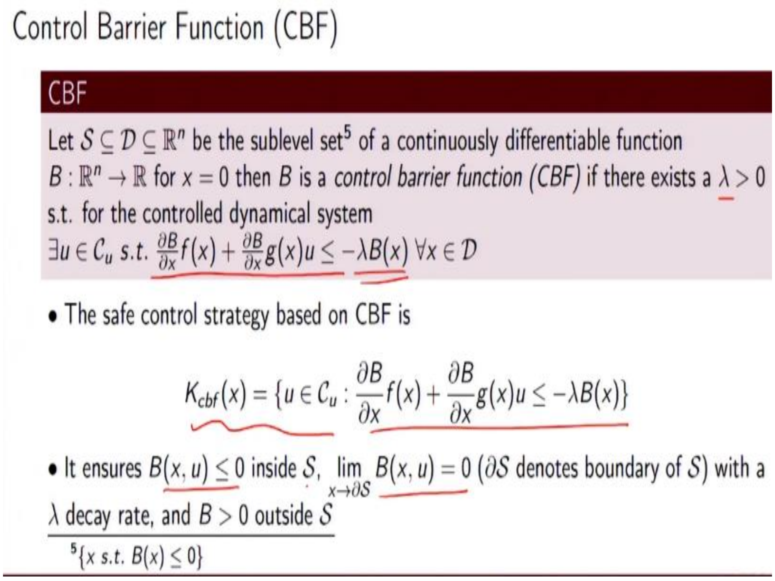

So this is how we define the control barrier function then right that well it is a function which just like the Lyapunov function will satisfy this condition but you see minus lambda B this is what it requires okay there exists this positive lambda so that this happens okay.

So if I want to ensure that my control value is such that it satisfies the safety requirement then my control strategy would be such that I should always choose u that this gets satisfied so that is I am able to smoothly control the value of the barrier function and keep it inside the safety region. So this is ensuring that this is negative inside S and it is in a limiting condition when I approach the boundary this becomes zero for with the with this decay rate right and outside it is definitely positive okay. So that is the trick and I hope with this most of the understanding is clear and then all it depends is we need to combine them.

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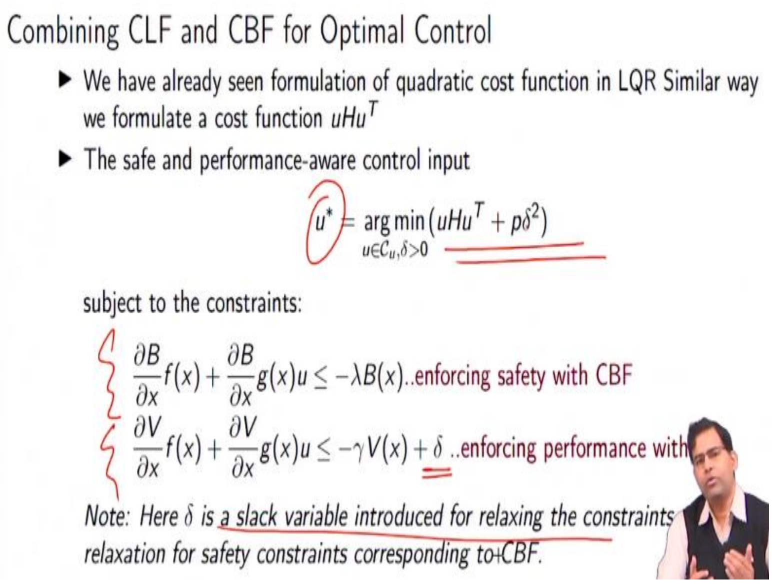

We have already seen that how to create the quadratic cost function in LQR. We similarly create this kind of a quadratic cost function again. So we say that well we what we want to do is we want to generate a u so that this summation is minimized okay. It is basically like the something like the older LQR thing right.

So quadratic term over the control and there is also the slack term. We will see what this slack term means we want to minimize this thing subject to constraints the first constraint is the barrier function the second constraint is the Lyapunov function. You see inside the Lyapunov function we add some small delta value which is also here. This is called the slack variable.

It relaxes the constraints because if you can if you can understand that in the Lyapunov function you I mean if I just force the Lyapunov function all the sublevel sets they are also invariant right. So I can actually relax the constraint there. But for safety no such reduction is done. We do not give any relaxation on the safety the moment I am approaching the safety boundary I must go back right. So there is no slack variable here I do not relax this constraint. But here I have this as a tuning parameter. When I am doing this control design, I can choose delta so that well finally here the Lyapunov function is giving me an exponential decay performance I can have some relaxation over it okay. So this is how we define the control and also make it safe here okay.

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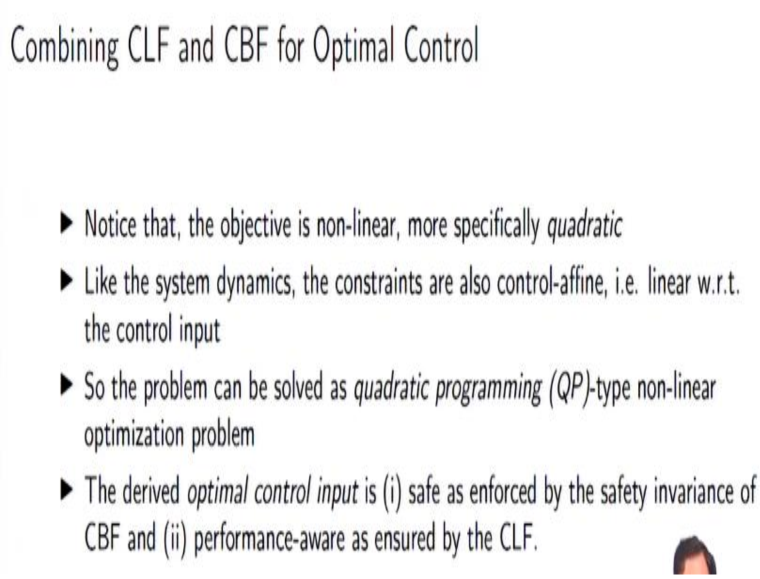

So the objective is non-linear and quadratic like you see and also you can use some well-known quadratic programming solvers to solve this non-linear optimization problem and safety is in I mean enforced by the CBF and the performance is enforced here by the exponential decay of the control Lyapunov function.

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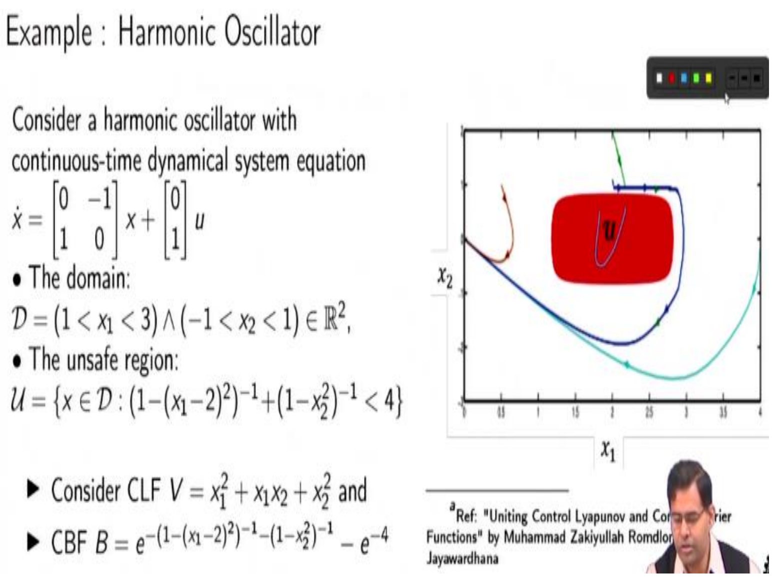

So if you take that harmonic oscillator example again and if you create these kinds of domain of the system if you give an unsafe set here. So if you see that well this is my unsafe region specification as given here let the this is my entire domain x_1 belonging to zero to three right like that well is somewhere here and for this system let us assume these two functions. This is my CLF and this is my CBF this has been taken from this example here and also our treatment of the control Lyapunov and control barrier function was taken from a paper by Samuel Coogan et al.

I think we have already referred that earlier if you see the references are right here. This is the reference by Magnus Egerstedt and others Aaron Ames and Coogan yeah Magnus Egerstedt Paul Tabuada ecc two twenty nineteen two thousand nineteen paper. So here we we actually see this example taken from this reference and we see that well if we consider this candidates Lyapunov function and barrier function for I mean here for this we can show that well how the systems trajectory is satisfying the given constraints okay.

So you can actually see that well the system is moving around and I mean if you do a control synthesis for this the system will be moving around by the way this is a capital I right because this is the this is the unsafe region okay and here we have the system moving around but it is never going inside the unsafe region as we see.

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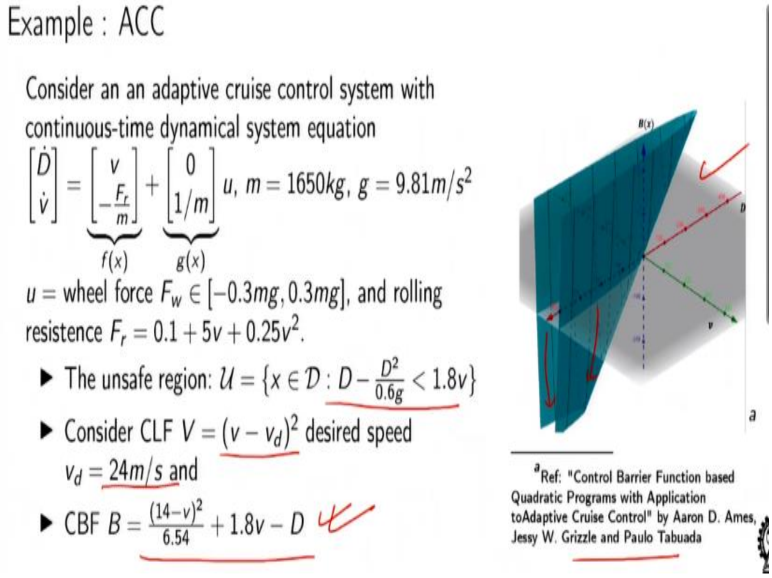

So let us take an ACC example a cruise control example usual set of equations with the control input being the wheel force that you want to give okay and also the rolling resistance equation is given like this. And let us say you have an unsafe region so that which is the unsafe region is specified like this okay and for this system let us say you create a Lyapunov function given like this v minus v_d square okay, where the desired speed v_d is given like this okay. And you have this as your CBF okay. All the usual parameters are specified.

Mass is given g value is given and u is the wheel force and f w this parameter is inside this range and also the rolling resistance is given by this equation. So this is from the from this example this paper actually you can just see. So what will you see that well if you plot the barrier function it looks like this. So it is like the front of a ship right a submerged ship this is the water body these are submerged ship right and here this you want to be inside this zero level downstream right okay.

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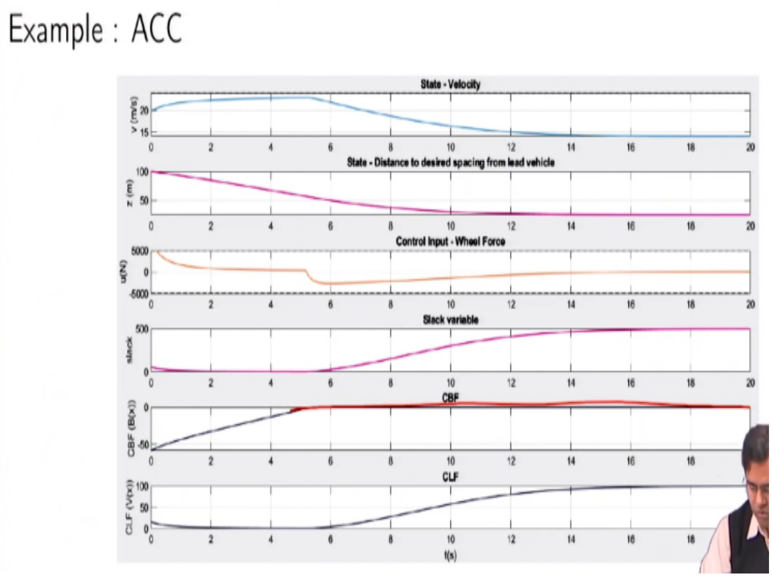

and if you if you do the constraint optimization that we discussed earlier you will see that well for this system if I am running the ACC using a controller which has been designed using a quadratic programming like we a formulation like we discussed here you see that well even. So if I plot the distance to the desired spacing that means.

What is the I mean let us say I have said that the spacing from the lead vehicle should be zero I mean it is not it is not ideal of course you see that it is really approaching I am almost reaching the lead vehicle here okay and I give a set speed I am reaching the target set speed here okay. And if you see here what happens to the CBF right. You see that the CBF is approaching the safe region but it is never crossing the boundary the safety boundary, it is never crossing the zero okay.

Similarly we have a plot for the other things also so well actually if you see this paper they have an implementation available you can download and check but during our tutorials and examples we will actually show you with inside our assignments that well how to solve this optimization problem and how to generate these plots. We will actually give you an example of that. How to do all those things you will be able to do them by yourself using suitable guidelines that I pro we provide you inside the weekly assignment. Thank you for your attention. We will end this lecture.