Foundations of Cyber Physical Systems
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Lecture – 41
Lyapunov Stability, Barrier Functions (Continued)

Hello and welcome back to this lecture series on Foundations of Cyber Physical Systems.

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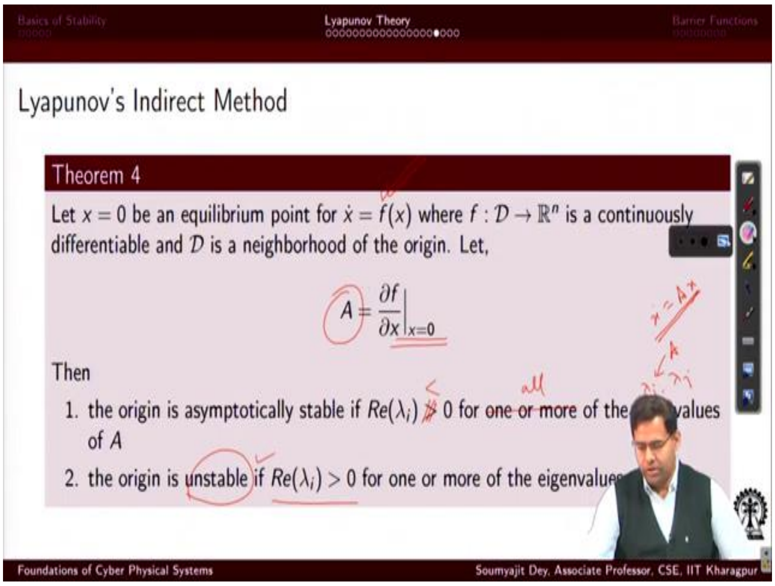

So, in the last lecture I believe we covered ah some of the Lyapunov stability theorems. And we discussed about the basic direct methods and also the indirect method. ah So, today we will start with the last theorem in this part of our coverage ah which is ah which is this theorem 4. Ah So, let us just directly get into the statement of this theorem. So, what this says is that ah like, like all our previous cases, we are considering this origin as the equilibrium point, for ah autonomous system which is just specified using this function f OK which is a continuously differentiable map with D is defined in this domain D.

Ok Now, ah we will do something new here which is like. Let us consider that we differentiate this function with respect to x in the state space. OK And then evaluate it ah at the origin OK and that gives us a matrix, A. ok. So, we will say that the origin is asymptotically, stable. Ok. If the, if we consider, ah that the real part of this I mean eigen value lambda i is less than 0 for 1 or more of the ah eigen values of A, sorry. ah This should be ah it is, of course, less than 0 for all of the eigen values of A. OK Ah So, as you can see that is just our basic idea of stability right what we say that well for the linear system let us say my system originally was x dot equal to Ax and for these to be stable. ah We want that well for A ah whatever are the eigen values.

Let us say lambda i lambda j whatever the eigenvalues for all of them. We will need that their real parts are negative. Right So, the eigenvalue may have a real and imaginary part and for all of them the real part should be negative. right So that was our requirement. What this theorem tells us is that well? ah This idea can be generalized we can consider any autonomous system and we can kind of linearized that system around the origin to obtain this matrix A.

And once we do that for this matrix A ah the same thing will hold. That means for this, as long as for all the eigen values of A we can show ah that we can actually see that if, for all the eigenvalues of a if the real part is negative we can say that system is asymptotically stable at the origin. ok ah And that would also mean the other thing that well if there exist some eigen values, if there exist some lambda i, for which ah the real part is positive then the origin is unstable. Right.

So, there is the idea. All we are saying is these are standard stability results. Right. That for all the eigenvalues they have to have negative real parts to be stable and if any one of them has a positive real part then it is unstable. We are saying that well that idea can be also applied to this domain of non-linear systems as long as you can obtain a linear matrix representation for that by by differentiating the system at the origin.

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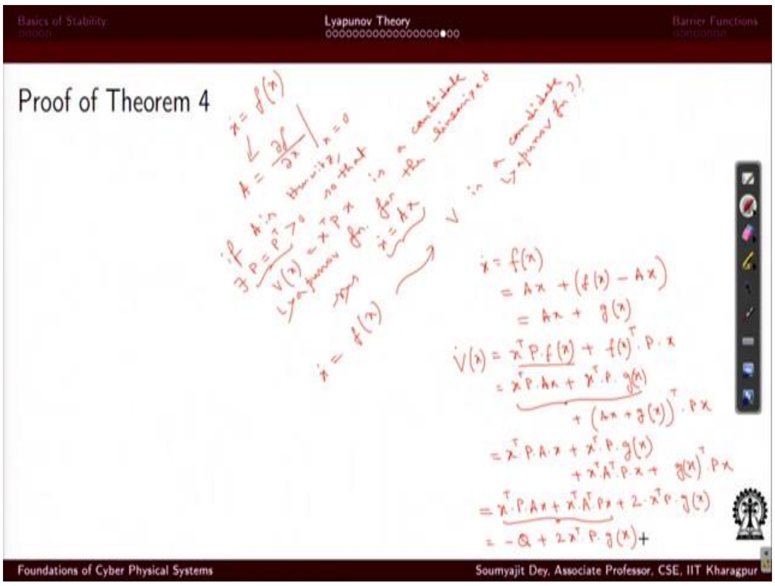

So, fine, ah let us try and create a proof, ah for this using our ah Lyapunov style of arguments here. Right. So, we started with this and then we said that well we linearized it. Right. At the origin and that gives me ah this representation A. right. Now, if A is Hurwitz, so, this is the linearized version and we can say applying our previous results that as long as A is Hurwitz ah then for this linearized system as long as we can figure out a P, I can always which is basically ah symmetric and positive definite. We can have this V equal to x transpose P x as a candidate Lyapunov function for this linearized system. Right. Now, what we want to show is that well, under what conditions this V can also be a candidate Lyapunov function for the original non-linear function f around the origin. Let us see. So, this is my original system and for this system we want to see that well if V(x) can be a candidate Lyapunov function. So, this is the question.

So, let us try one thing. So, this is my original system. Now, let us break it into two parts. The first part being the linear part and this is the residue. So, let us call this residue as g(x). ok. So, for this V if I take a derivative now, a time derivative, let us see what we get. So, x transpose P and then you have x dot so that becomes f(x) here right because x dot is f x plus. What do you really have? You have x dot transpose. Right.

So that will give you f(x) transpose right, times P x. So, basically, instead of A you are having f(x) here right ah while doing this ah derivative of the Lyapunov function. So now, let us try to bring in the linear and non-linear parts here. So, I can write these two together gives me this first term right and then I have f(x) transpose Px. So that is what we have. So, this transpose can go in. Now, let us see what is g(x) transpose. I mean we should be able to write this as like this.

So, if we apply the matrix Lyapunov equation, we can write this as a minus Q to g(x) that is what you have. right Now, let us observe some property of g. So, what we can show is something like this.

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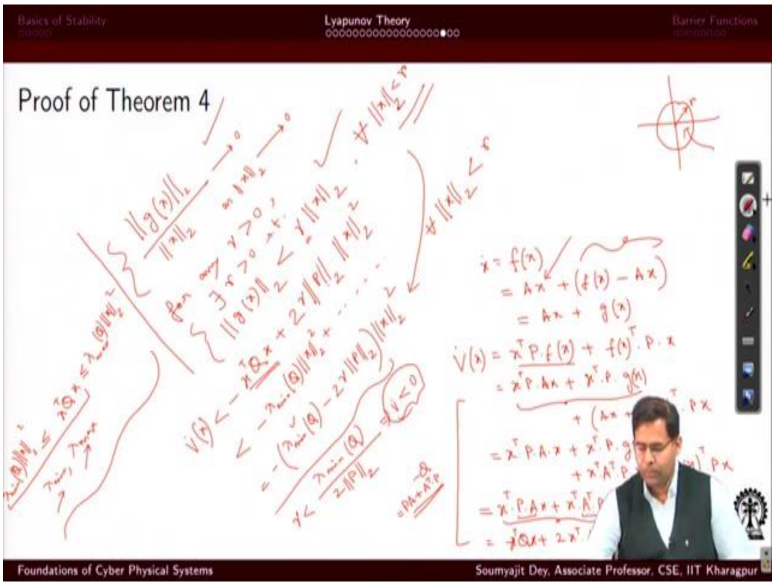

So, let me delete this part here, the norm of g(x). So, second norm to be more specific by the norm of x. This will tend to 0, as the norm of x will approach 0. So, the point is I mean this is obvious because if you understand what is Ax from the definition. So, Ax was the linearization of f(x) itself around the origin. right So that means when ah x is approaching the origin, the function g(x) is going to be infinitesimally small.

And that is why, ah in the limit condition, when I have x approaching 0, this ratio would also be approaching 0. Ok. So, what we can say in such cases is that any positive scalar for this, there exists some other scalar r which is also positive, such that this norm. So, this is it. So, for any scalar we can find another scalar r which is sufficiently small. So that the moment I have x less than r this value of g would be bounded upper bounded by gamma times x.

So, for any such gamma, I can find a suitably small r so that the moment the value of x is approaching 0 and it is already less than r. I will have the value of g(x) small enough to be upper bounded by this gamma times x. So, I hope, I hope it is clear that we are trying to linearized around the origin. So, the function is approaching, the function f(x) is approaching towards x and the residue g(x) is also approaching the origin.

And when x is sufficiently small ah we can always figure out some upper bound, some lower bound, some upper bound like this that when x process that then I can bound this value of g(x) by this gamma times x. ok So, this is ah stand coming from the continuity of these functions and what we can show as such is well, we can then write since I am able to bound g(x) by gamma x_2. So, I can then replace this g(x) by this upper bound.

And I can then say that well, derivative of x which derivative of V(x) which is less than which is equal to minus Q plus this containing g(x) should be less than the same thing, where g(x) is replaced by the upper bound right. Sorry here, it is x transpose Q x. right Because the matrix Lyapunov equation what it will do is, ah, if you remember ah if it is, if we apply it here, it was minus Q equal to ah this P A plus A transpose P. Right. So that is why it gets applied.

So, this part becomes, this part is what becomes Q and x transpose and x remains. So that is minus x transpose Q x. So, this remains here and this g x will get upper bounded. So, what we get is, plus 2 and you are replacing this g x with this upper bound. Right. So, the upper bound being ah some gamma times this. Right. So, this will technically give you the second norm square. So, let us understand what just happened. So, this P was there. Right.

Now, g x gets replaced by the upper bound right so and that upper bound is, I mean, gamma times x. right So ah then ah essentially, this is a scalar right, so, you will have ah this 2 and x transpose and the Ax together will give you the norm and you have the value of P here in a scalar form which is the norm. right. So, ah this is how you get the upper bound but you also have ah some other facts here which we can apply. Ah

So, we will, we will apply this well known, result from matrix algebra. Which is that we can show that this thing x transpose Q x this quadratic form, ah so, one thing let us understand this will eventually give you a scalar. Right. So, this has a well-defined upper and lower bound. And it is bounded by, yeah so, given this quadratic form x transpose Q x if lambda mean of Q and lambda max of Q represent the minimum and maximum eigen values of this matrix Q, then for x transpose Q x we have a well, defined upper and lower bound. The lower bound is x min the lambda mean times Ah this excess norm Ah so and the upper bound is lambda max times the x axis norm square. ok I mean on both sides is lambda, mean times x norm square and is lower bound is lambda mean times x norm square. Ok. So, ah we can actually apply that ah constraint and we can actually show that well ah if this is true, then if I apply this part here and if I replace this with an even smaller value. right then Then I can say that this is even less than yeah and these all holds for this condition. That is for all values of x so that the norm is less than r. So, once we are inside this r radius ball, we bring x inside this r radius ball. We have this condition holds that that holds because of this because as we approach more towards origin, this this component will approach 0. Right.

So, this holds and from that we can write this ah. Basically, we are replacing, we are applying this condition to this. And then we are applying this lambda mean bound here and we get this condition. Now, what does this mean? That, this means that I can actually make a choice of gamma here. I can actually make a choice of gamma to be less than and we can say that well as long as this happens then this quantity is greater than this quantity.

So, this is all negative. And we have a guarantee that the derivative is less than 0. That is what we are targeting. Right. And so that the derivative will be negative definite and we can say that the non-linear system is a kind of asymptotic or is stable. ok so ah. So, this gives you the first part of the proof. The other part that when the origin is unstable because at least one if one of the 1 of the ah 1 of the eigen values have a positive real part.

That will that that proof requires ah ah instability result. So, we are not going up to that. So, I hope this much is clear. We are able to show that well as long as we have linearized this non-linear system. And the linearized part is actually satisfying the stability condition that A is Hurwitz with that means all the eigenvalues of negative real parts. Then we can show that well ah the system is stable as long as this A continues to be Hurwitz like this.

So, we have applied ah certain tricks here, for example Ah we have applied this continuity condition. We have applied this ah inequality using, ah I mean inequality of which relates the quadratic form ah with eigenvalues. And with that we can, what we can effectively do is? We are able to calculate a value of this parameter gamma which tells me that well for what choice Ah this this P should be negative. Right.

So, accordingly then we are able to state that will in case we have linearized the system. Ah. And after the linearization, the matrix that we obtain, we have all the eigenvalues of negative. Then the system is stable around the origin. Ok. So, fine ah that is about our proof of this theorem.

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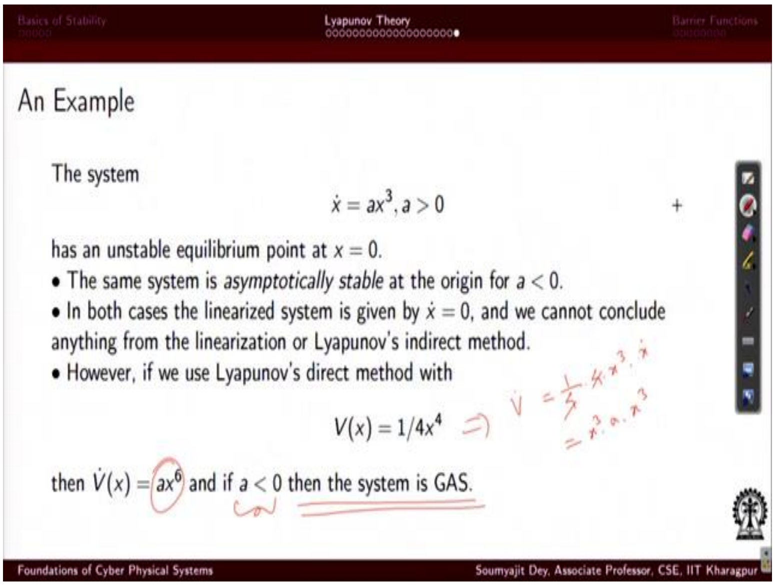

So, let us, through this example here so, suppose this is the system and this system has an unstable equilibrium ok ah at x equal to 0. The system is stable at the origin, ah for this value of A ah to be less than 0. So, if A is positive ah then we have this unstable equilibrium but if A is negative, the system is stable at the origin. I mean intuitively, you can see that actually. Right. I mean and in both the cases when you have A greater than 0 or A less than 0. Ah

If you try to linearized the system ah what you obtain is an equation like this, like x dot equal to 0. So, this this is the situation ah where you cannot conclude anything from the linearization or the Lyapunov indirect methods. But if you use the Lyapunov direct method, so, what does the indirect methods effectively tell you? They try to provide you with an idea that how to choose the candidate Lyapunov function.

Like in theorem 3 we say that well, for a linear system this should be your choice of Lyapunov function and this inequality. The matrix level of function in the matrix Lyapunov equality must hold etcetera, etcetera. And in the linearized case, we show that well, for if you linearized and if your matrix is Hurwitz then well the same Lyapunov function will actually guarantee. Ah

I mean the same similar structure of Lyapunov function ah with suitable choice of parameters will actually guarantee that the system is stable. So, those are like directions which tell you that how to pick up the Lyapunov function. So that is why it is an indirect method. The direct method is that you directly go and choose the Lyapunov function. So, we see that this is a system where we cannot apply the indirect methods. Ah

That means I we cannot apply to linearized. We cannot also and this is not a linear system that I just directly apply the matrix Lyapunov equation. Right. So, those methods do not work because if I order if I linearized then I have this and also this is a non-linear system. So, the retirement matrix Lyapunov equation is not directly applicable also. Ok. But if I if I use the direct method and I somehow just arrive at this choice of this is a Lyapunov function.

Then you see that if you take a derivative here. Ok. ah So, let us see what you get? So, this is basically, a derivative of x right. So, times x dot, right sorry. So then you have x dot here, so that is A times x to the power 6. Right. And if A is negative, ah you can actually show ah that will this system is globally asymptotically stable. Around this unstable equilibrium point I mean it was unstable for this. ah

But I mean but when A was positive but ah we took A as negative. And then we said that we will take this as the function. And then we get this as the derivative. And if you take the second case that A is less than 0. Then we can show that well, since this is my function and this is the derivative and A is negative. We can actually demonstrate here that this system is a globally asymptotically stable at the origin ah for this choice of A being negative here. ok so.

So more or less that is about our ah coverage on Lyapunov functions and let us now move over ah to something different from this. So, ah ideally, whenever we have been talking about Lyapunov function, it has already it has always been about the stability of the system. Right.

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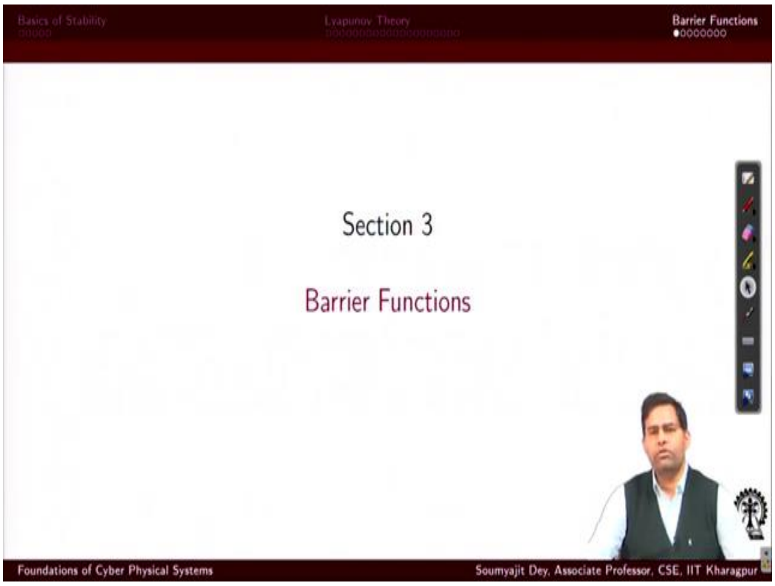

But ah let us understand that will ah while stability is an important property, ah we also need to talk about the safety of control systems. Ok. So, when we talk about safety of control systems, we use a notion called barrier functions. So, this is what we will study in the next lecture. Thank you for your attention.