Foundations of Cyber Physical Systems
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Lecture – 40
Lyapunov Stability, Barrier Functions (Continued)

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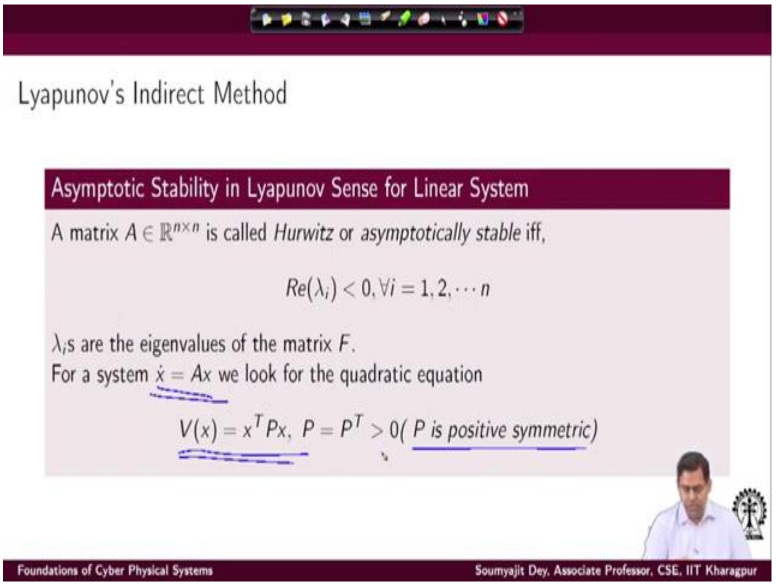

Welcome back to this lecture series on Foundations of Cyber Physical Systems. So, in the previous lecture we were discussing about Lyapunov Stability Methods. And so, we have covered this Lyapunov of in the direct methods. And now, let us come to Lyapunov, indirect methods. So, through which this will be analysing, asymptotic stability for linear systems. But we already know that well how we analyse for linear systems?

And we will just check that well that also nicely corresponds to a Lyapunov style argument. So, for a system matrix A which I can create for a linear system. We know that it is called Hurwitz or it is asymptotically stable. If we can compute the eigenvalues of this and we can see that well, the real parts of those eigenvalues for all of them they are negative. So, that is when we say that the system is asymptotically, stable or which stable.

So, well since Lyapunov style arguments cover for all classes of systems, they must also be covering for linear systems. And as it happens it is it really works. So, we can show that well for linear systems we can construct a Lyapunov function in a structured manner like this. So, suppose you are given this linear system in the form x dot = Ax you have this state space representation.

And for this you can I mean you want to create a corresponding Lyapunov of function and try to analyse stability. So, let us assume the function is of this specific quadratic form x transpose Px. Now, this choice of matrix P will be something specific, so, P will be a symmetric matrix P is equal to P transpose and P is also positive definite. That means it is positive definite and symmetric.

That means P all the components of P are strictly greater than 0 is positive definite and it is symmetric also so, P = P transpose greater than 0.

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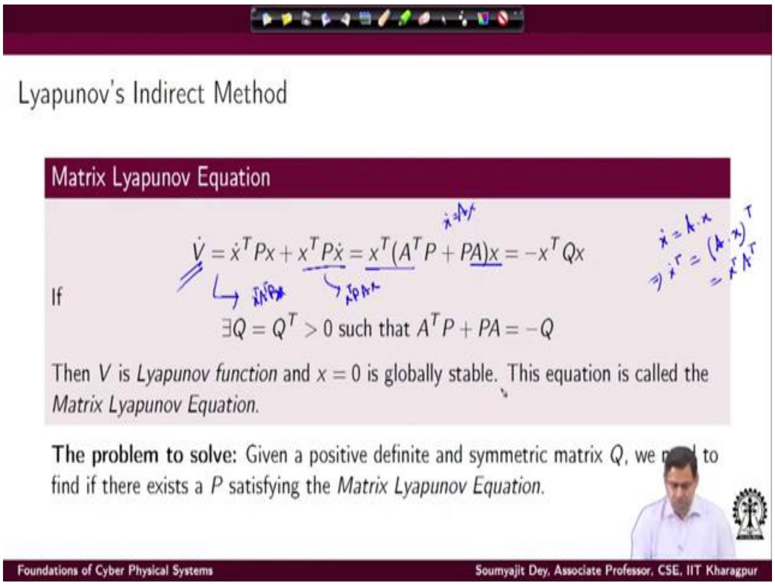

So, this is the specific class of functions we will be looking for and so, assuming that such a p exists and let us say this is my form of the Lyapunov function. Now, let us go ahead and differentiate this because our requirement is, if you recall that this derivative has to be negative and all that. So, our first interest will be to kind of differentiate this function. So, if we differentiate what we will have is fine.

We will first have a derivative of x here, followed by with respect to time. So, you will have x dot transpose Px and then again you will have a derivative of xA with respect to time. Because P is a constant matrix anyway, so, x transpose Px dot. So, if you substitute x dot = Ax in both places. So, what do you get? So, here it is so, x dot transpose, is x transpose A transpose. So that is what you have in the first term.

And in the next term I mean here you just have x dot, so, you just have P followed by Ax. So, what I have from this is? Essentially x transpose, A transpose Px and here you have x transpose Px. So that is why you can again create this quadratic form by taking x transpose common beforehand and x common after hand then in between you will have A transpose P + PA. So now, let us observe our requirement.

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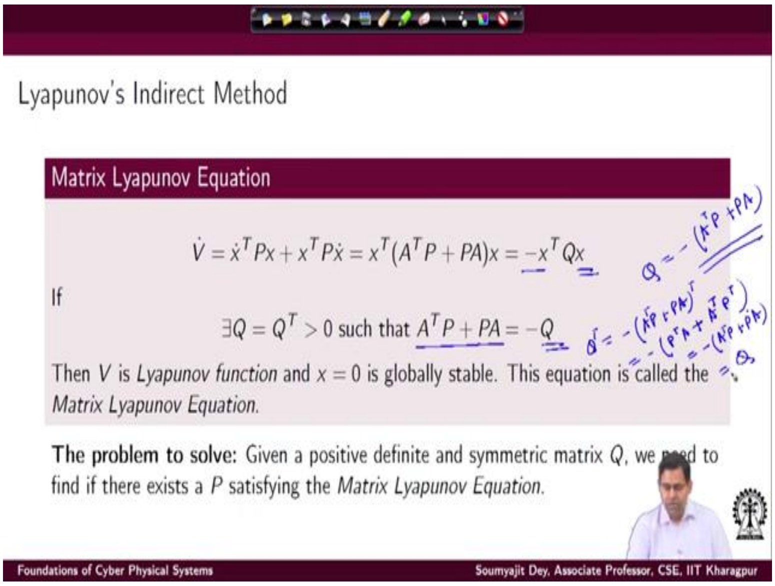

Our requirement is V dot has to be negative. So now, let us write it in this form and claim that well, I should be able to write this as minus of x transpose Qx where Q is equal to I mean minus of A transpose P + PA. So, the point is, as long as I can figure out a positive symmetric matrix P for which if I write a transpose P + PA. The resulting matrix is I mean, is kind of definable like this.

That means it is negative definite or I can write it as –Q with Q as positive definite. And of course, I means by this form I can say that Q is also symmetric. Because if this happens, if you take a transpose here, Q transpose. So then what do you get? The transpose goes in and acts on the first component. So, P transpose T transpose A T transpose then well, you just get A here. And then the next one A transpose P transpose.

And since P transpose equal P by other assumption then you have this minus of A transpose P + PA because P transpose equal P. So, if you just apply that this is what you have and then well this is again same as Q. So, by this design Q will be also symmetric and by this design we are claiming that well as long as this thing is a negative definite. We can claim that the system is a global system, is a system is GAS.

Because I have been able to find Lyapunov function which satisfies all the requirements. So that is why we call this as the matrix Lyapunov equation that means you start with this assumption that V(x) can be expressed as x transpose Px where P is a positive symmetric matrix and then if you take the derivative you apply at this, you arrive at this nice form. Where you have x I mean the derivative as x transpose followed by A transpose P + PA times x.

And which you can write in this nice form and as long as you can show that this A transpose, P + PA can be expressed. I mean, of course, as long as P is symmetric, A transpose P + PA will be symmetric that is a given. So, as long as there exists, this Q = Q transpose greater than 0 such that this thing happens. We can say that the system defined by A is globally stable at x = 0.

And this equation is known as matrix Lyapunov equations. So, the idea is, if you are given a system, my matrix A which is Hurwitz stable or asymptotically stable. We, I mean it must be satisfying this matrix Lyapunov equation because if it is stable then it must be stable in the sense of Lyapunov. And if it is stable in the sense of Lyapunov we should be able to prove that this thing happens. That if this is the derivative.

I mean well of course, this thing is subject to assuming that the Lyapunov function is in this quadratic form. And if I take the assumption that well it is in this quadratic form. And if I am able to show that this P and Q exists and they are connected by this relation. Then also I am able to show that well this system is stable. So, the problem to solve in this case is that suppose you are given a positive and symmetric matrix Q.

We need to find that there exists a P that wise, this matrix Lyapunov equation. The question is how do I find that? So, you are given this Q which is positive definite and symmetric as you can see that the Q is positive definite and Q is definitely symmetric. So, Q is Q transpose and since it is positive definite. So, negative of Q is the derivative of V which becomes negative.

So, as long as I can figure out this P for which this thing happens that a transpose P + PA is symmetric. And I mean it is negative, definite or the corresponding Q is positive definite. Then we can say that well I have a P that satisfies the matrix Lyapunov equation.

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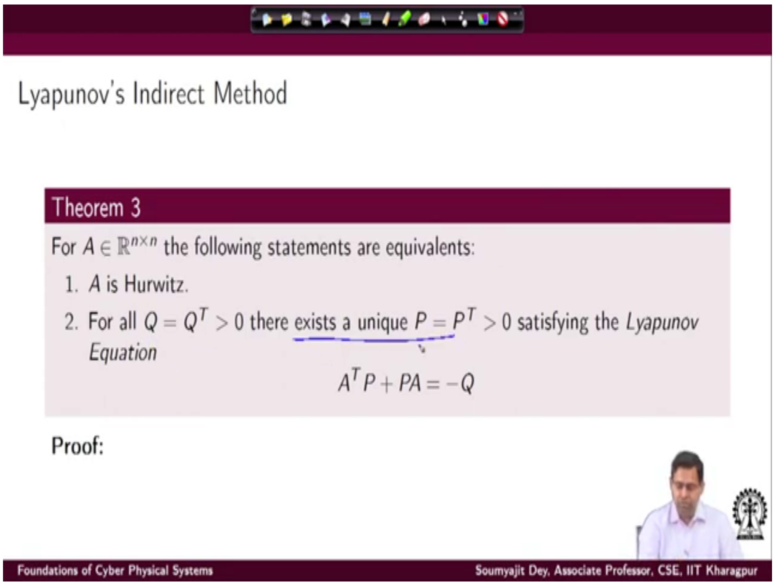

So, the question is how do I know that such a Q exists such a P exists? So that is what this theorem says. The theorem says that well if these two things are equivalent, if I get a system matrix say A being her which stable will guarantee that for all positive definite symmetric matrices Q. There will exist you make positive definite, symmetric matrix P such that they together satisfy this matrix Lyapunov equation.

So, note the thing it says that if this is true then this is also true and it is equivalent. That means, if the second clause is true, the first clause is also true. So that means what does the second clause really mean? It means given A I should be always able to identify this P and Q. I can choose any Q which is symmetric and positive definite. And for that I can show that there exists I mean this P which is also symmetric and positive definite.

So that means this PQ, PA will satisfy this equation. So, let us think that will how I can go about proving this? So, we will just do one side of this. We will assume that well, A is always stable and we will see that well how this can be computed? That means we will show that well it is indeed the case that such a unique P exists. So, notice this thing that P has to be unique for a given Q for all Q we have a unique P.

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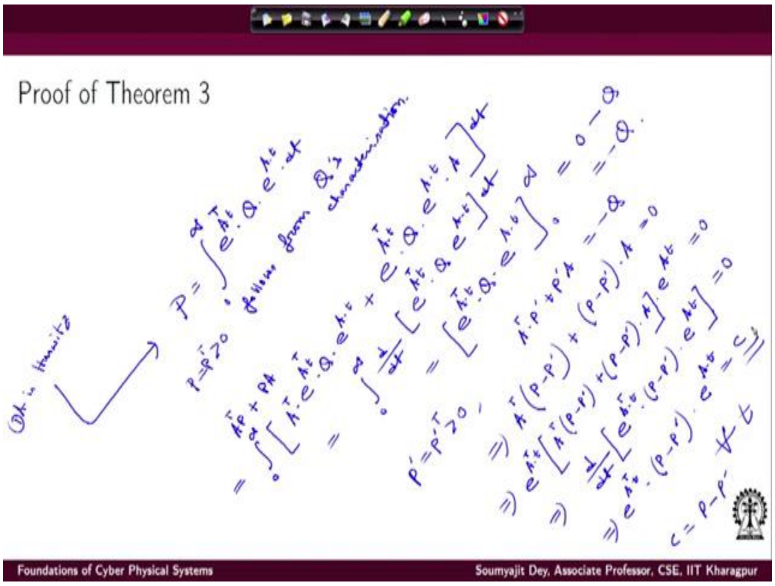

So, the point is, you are given the Q and we need to figure out the P. So, let us first propose a P in this case. So, we are assuming that A is Hurwitz and you are given the Q. We will be trying to arrive at a unique P so that concludes one side of the proof of this theorem. So, what we will do is? We will propose a solution that given Q P can be this. So, let us assume P is given by this integral 0 to infinity e to the power A transpose times t times Q times e to the power A times t derivative.

So, let me just write this properly again. So, we are assuming this as a form of P and let us try to see that where this satisfies all the requirements that P is symmetric, P is positive definite and P is unique and P satisfies the matrix Lyapunov equations. So, by construction we can see that well if Q is assumed to be symmetric. Then well this thing definitely I mean, if I take a transpose P will also be symmetric.

And if Q is positive definite then P will also be positive definite because you are integrating here like that. Now, observe one thing this integral will converge only if A is Hurwitz. Because if A is Hurwitz then we know that when I integrate this. And I mean this the values of A are kind of decreasing with increasing of time. And then I mean this integral will have convergence.

If A is non Hurwitz you do not have damping and then this thing will not converge like that in that way. So, that is it so that actually tells you that how the other claim that A is Hurwitz relates to this, like only if this happens then P can actually exist. So that is how whatever we are doing is linking up to my original claim that we are assuming P as Hurwitz. So, if we now write the matrix Lyapunov equation and put the value of P here.

So, you are just pushing P into this matrix Lyapunov equations format and you will get this integral. So, this you can see is nothing but it can be written as I mean this is like a derivative of this term. Yes, so then you have the integral practically solved, so that is what you have. So, if you now do this up so, since A is Hurwitz you have at infinity, these are 0. And at 0 you have both of these as a I mean both these exponential times become 1.

And so, you have Q so, this is equal to –Q. So that means this choice of P will be a candidate solution, for this matrix Lyapunov equation. But the question is this P unique. So, let us assume that there exists another form let us say that is P prime. So, P prime = P prime transpose greater than 0 and both P and P prime satisfies this. So, just like this is satisfying this thing I can say that A transpose P prime + P prime A is also equal to – Q.

So, if we just now do a subtraction from for both this matrix Lyapunov equation forms. So, you have a transpose P – P prime + P – P prime times A = 0. On the right hand side both things get cancelled out. So, we can now say if we just multiply on both sides that is what we have and then this is nothing but derivative. So, from this, what we can conclude is? So, this is the derivative is 0.

So that means this thing e to the power A transpose T times P – P prime times c to the power A times t equal to some constant. But see this is going to be true for all times. So, this will also hold for some time t = 0. Now, if you see that then if I put t = 0 then this thing becomes P – P prime. So then it tells me that this because this is true for all times. If this is constant then only the derivative is 0.

So, essentially with t = 0 I get the value of the constant as P – P prime and that means every time for all other t actually we must have this thing equal to P – P prime.

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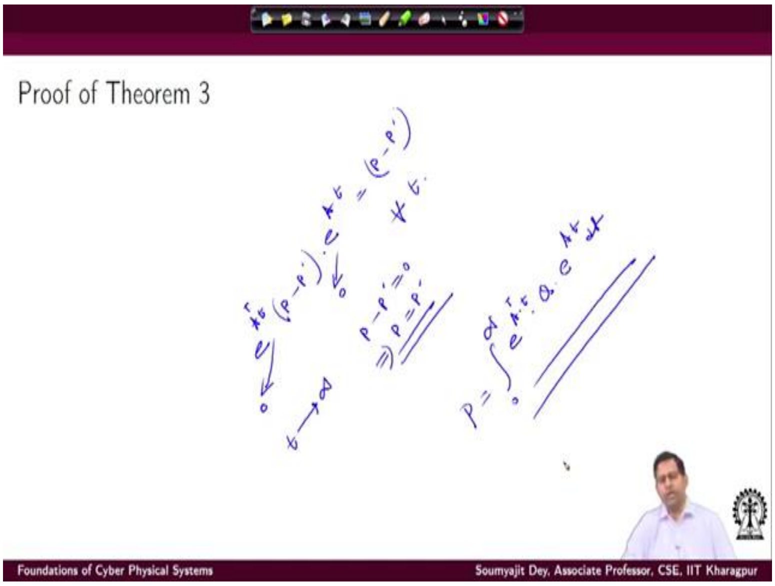

So, if we write this then what we have is? In general it is coming up like this since that constant value I mean if I put t = 0 then we simply get this to be constant so that is what we have. So, if you see, if I also let t tending to infinity since this is true for all situations for all t. So, let t tends to infinity again using the premise that this is Hurwitz we will have both these terms tending to 0.

So that means eventually, I will get P – P prime = 0. That means P = P prime. So that kind of proves to me that well this value of P that we have computed through the original integral. So, this is what we assume P to P. Let me just rewrite P was assumed to be having this nice form, so, this is like unique as long as A is Hurwitz this is going to be unique. And this form of P will satisfy the metric Lyapunov equation.

So that means given any Q which is positive definite and symmetric. I should be able to compute P using this for expression. And then I must be getting a P which is also positive definite and symmetric. And this P Q pair will satisfy the matrix Lyapunov equation. The theorem 3 is converse that means the fact that these wholes proves A is Hurwitz we are not doing it for the time being. Fine, let us end it here today. Thank you for your attention.