Foundations of Cyber Physical Systems
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Lecture – 39
Lyapunov Stability, Barrier Functions (Continued)

Welcome back to this lecture series on Foundations of CPS. So, in the previous lecture we proved Lyapunov direct method. And so, let us see some application of this method here.

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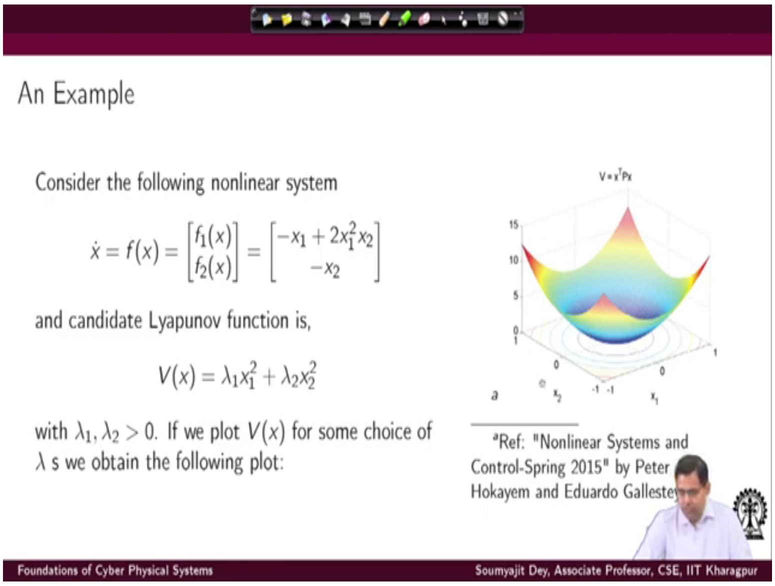

So, if you remember before doing that proof we were talking of this example of our system function a non-linear system, as you can see. And we assumed a candidate Lyapunov function from there and we showed that well how the function will look like? right If we plot it against x_1 and x_2 in this plane. Right. And here we are having this level sets that means the values of x_1 and x_2 for which the function V will will be actually getting evaluated to some constant. Ok.

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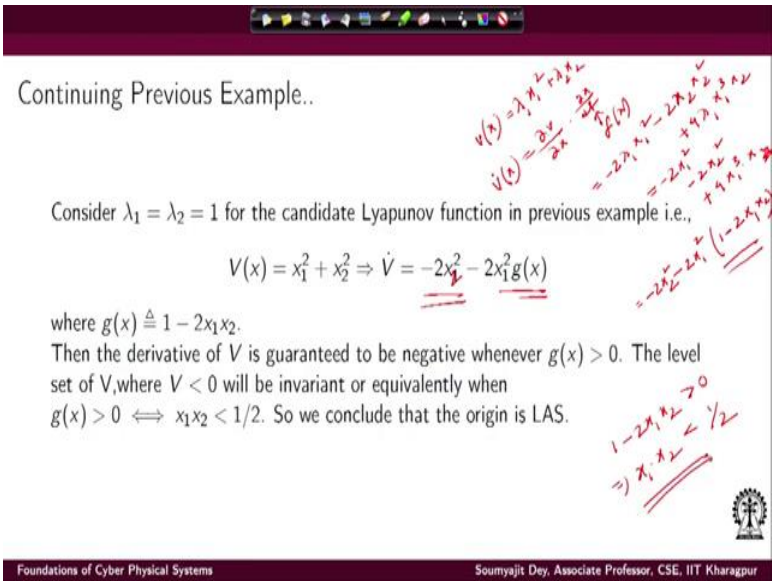

So, let us come back to that example. So, assume in that function that λ_1 and λ_2 is 0. So, V(x) is simple $x_1^2 + x_2^2$. If you take a derivative, so, you have so, with respect to time then you have well this. So, if you remember what was your x?

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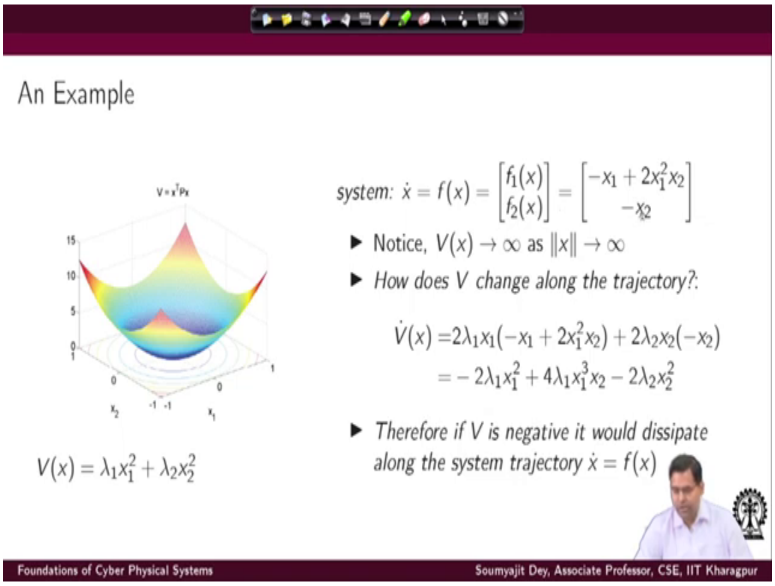

It was given by these systems right $-x_1 + 2x_1^2 x_2$ and $\dot{x_2}$is given by -x_2. Right.

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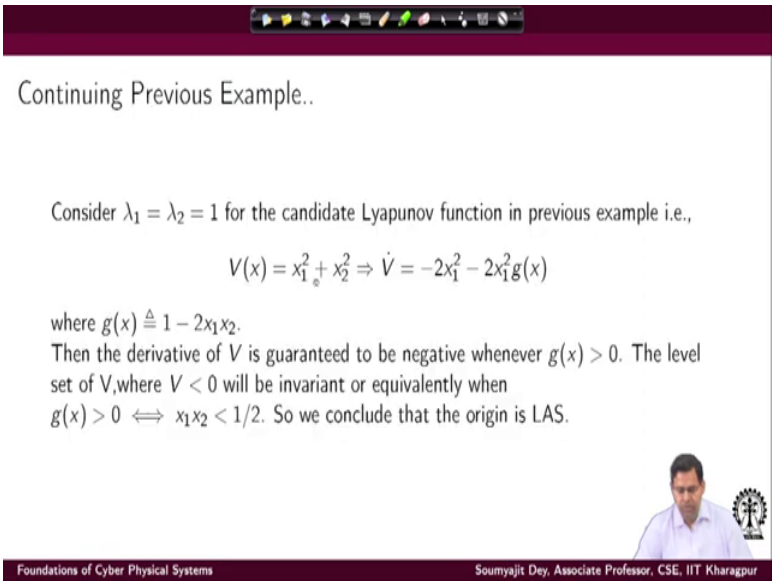

So, ah when you take the derivative, ah it will be basically you have to take the derivative with respect to x followed by ah the derivative of x and x_1 and x_2 both with respect to time. Right. So, ah if you if you calculate that ah you will you will arrive at the derivative of V as $-2x_1^2 - 2x_1^2 g(x)$. ok Now where g is given by this $1 - 2x_1x_2$. ok So, ah in general, if you are doing this for the original Lyapunov function.

So and with λ_1 λ_2 being there you had which is your f(x) and you already have this vector form for f(x). Right. So, if you compute this derivative here, so that is what you get. And then if you put λ_1 and λ_2 ah as 1. all right. So, this is what I can write. Right. So, eventually the derivative if I put λ_1 and λ_2 as 1 it is $-2x_1^2 - 2x_2^2 + 4x_1^3x_2$. Ok

So, ah yeah this is x_2 here. So that is essentially minus 2 x_2 square minus 2 x_1 square times 1 minus this. Now, why are we writing it in this form? The reason is see my target is to show ah that V dot is negative. So, this term being a square term with minus so that is negative. Right. And that means as long as I can say that this g this part g(x). So, let us call this as g(x) Right. So, I I have minus 2 x_2 square minus 2 x_1 square times g(x) right where this is $1 - 2x_1x_2$.

So, as long as I am located somewhere where this is ah positive that means x_1 times x_2 is less than half right. As long as this happens Ah V the derivative with respect to time of V is negative. Right So, ah we we see that will ah if I come, if I if I if I take this region around the origin. right ah, So, I I can show that the system will be ah locally asymptotically stable around that region. ah As long as I have this condition and getting satisfied. Ok.

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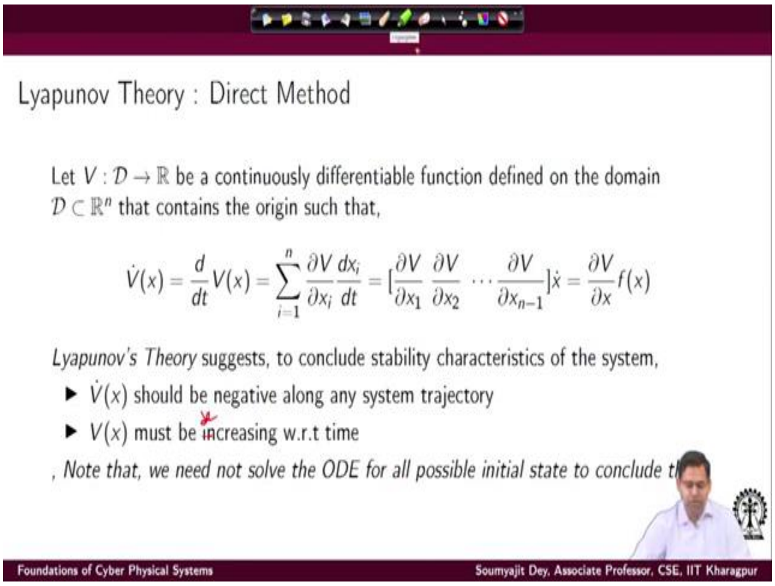

And one small correction here so, in this, when we were deriving ah the Lyapunov function, please note that this is decreasing with time because the derivative of V is with respect to time is negative. Fine. Ah So, we have seen that well how early Lyapunov function can be assumed. Ah We have to so, if you have to apply Lyapunov theory, you have to be smart about choosing this function like we see in this example.

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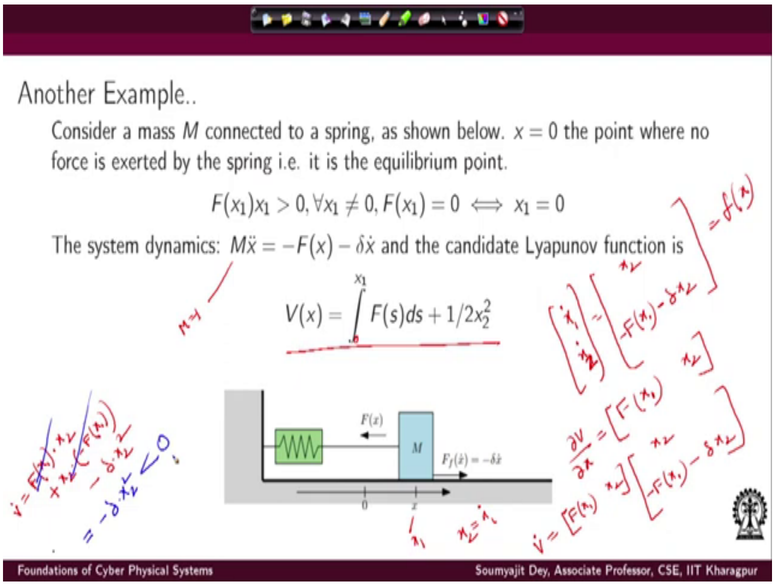

Now, let us take another example ah from the same material we have been covering here. So, we have a spring mass system, ah of mass M. right. It is connected to the spring ah and x equal to 0 is the point when no force is exerted by the spring ah since by this spring. Right. So that is the equilibrium point. And otherwise what you have is this, right. These are the these parameters of your spring mass system ah right ah so, the force that you have ah here is given is defined by this. Ah

That F of ah x dot is a kind of minus delta x dot. Ok. So, for all ah positions ah which is not the origin. ok ah You have $F(x_1)$ equal to 0, ah implying x_1 equal to 0 and otherwise you have ah this thing happening. So, when you are at the origin, there is no force and whenever you are not at the origin. Ah. The spring will be exerting a push or pull force on the mass air and the amount of this force is given by this Ah um this F(x) here. Right. So, if you are trying to derive the dynamical equation of the system. Ah

The acceleration of ah this mass ah x double dot would be divine defined by this total force that you have which is ah minus of f(x) minus delta x dot and so, fine. Let us lets Let us just go about defining the state variables of the system. So, ah if we just assume that let us say for simplicity, ah m equal to 1 with the 2 state variables x_1 and x_2. OK. ah So, x 1 is basically ah this I mean the deviation from this origin is given by x and that is my x_1 and x_2 is nothing but x_1 dot.

Because it is a I am eventually going to compute x_2 dot which is the acceleration. So, I will need these 2 state variables x_1 dot and x_2 dot. So, x_2 is speed, x_1 is the displacement. So, x_1 dot is nothing but acceleration x_2 ah sorry ah this displacements variation is nothing but the velocity x_2 and x_2 dot is the net acceleration here assuming ah the for inside the force equation. M is being set as 1 so, this is what you have.

So that is pretty much your definition of the system F x. So, for this system ah we consider this as a candidate Lyapunov function.

$$V(x) = \int_0^{x_1} F(s)ds + 1/2x_2^2$$

So, if you see it is defined as an integral from 0 to ah this state variable x_1 of ah sum this function F OK plus half of x_2 square. Ok, So, one thing to note if you see here, I mean this is this is going to be always positive. So now, if you do ah derivative of this function with respect to x, so, ah there would be two components. right ah.

So, in one you will have F because it is basically an integral of a F. right. So, if you take the derivative, it will be F which is defined over the first component x1 because F is defined over x_1 only. And the other component will will be x_2, because as you can see, it is half of x_2 square. So, derivative it is 2 x_2 ah the 2 and 2 will cut I mean it, they will cancel each other and you will generally have this x_2 in the second component.

So that is your derivative with respect to x. right So now, if you take the time derivative. So, you have $F(x_1)$ and x_2 I mean while taking this this 2 square cancelled, so, you have only x_2 and this thing here. Right. So, if I write it here multiplied by x_2 you are just doing this matrix multiplication here and then you multiply this. So, it is basically minus delta x_2 square. Now, you can see that these guys are again cancelling each other.

It is F x_1 x_2 and minus of F x_1 x_2 F x_1 times x_2 and minus of F x_1 F of x_1 times of x_2. So, what you have is minus delta x_2 square, right. So that is a negative quantity, so that actually proves that the system is stable and that is about it. You can You have actually been able to prove using this Lyapunov function that well. I have taken this candidate Lyapunov function in a clever manner. So that I can show that ah the derivative, the time derivative of this function is negative.

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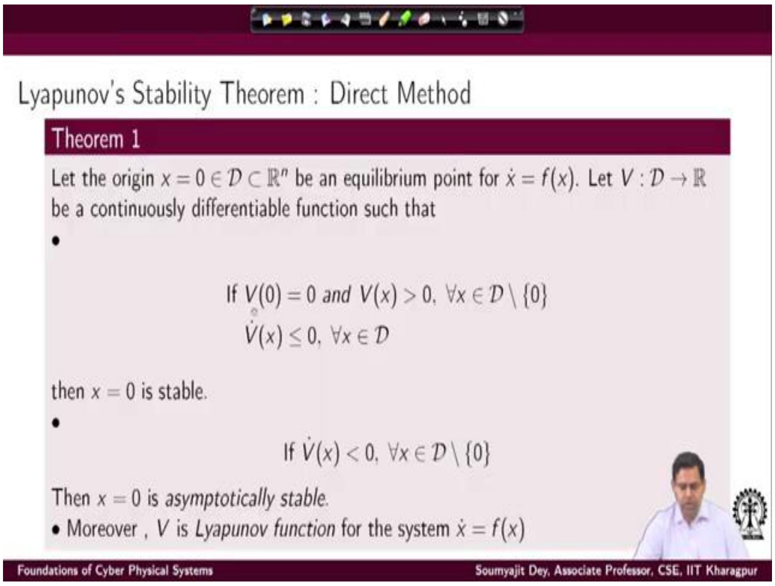

But also, ah let us recall Ah there were some other conditions ah does this choice satisfy the other conditions. So, the other conditions you had was well ah V(0) is 0. Right.

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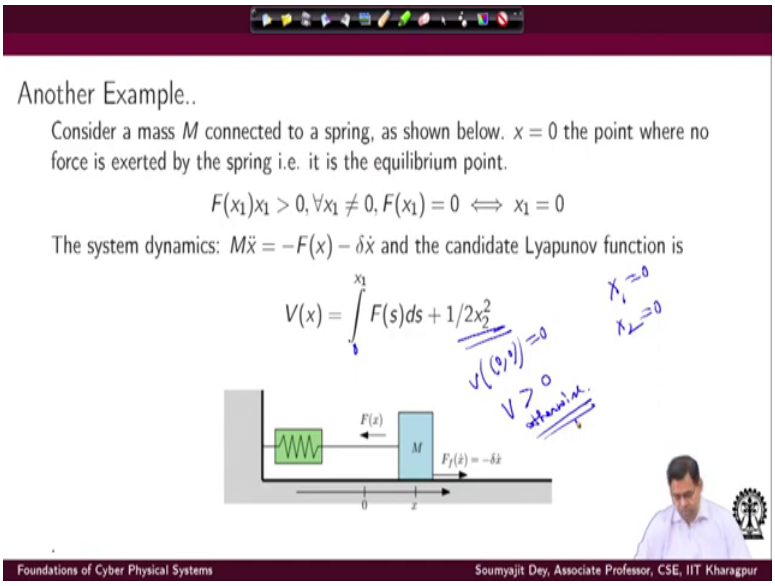

And otherwise everywhere the function is positive. So, if you see that the function is an integral plus a positive term, so, as long as x is non 0, this is going to be positive and if you have x_1 equal to 0 um I mean, if you have the If you have the or x equal to 0 that means you are at origin, with x_1 equal to 0 and x_2 equal to 0. So, this is 0 and this integral is ah I mean. Its, So at a point right so that 0. right ah So, you have V at x at the origin to be 0 and everywhere else Ah V is positive.

And we have shown that well ah the I mean V will be decreasing, ah everywhere In the I mean, apart from the origin. Right So, we can say that the system is stable.

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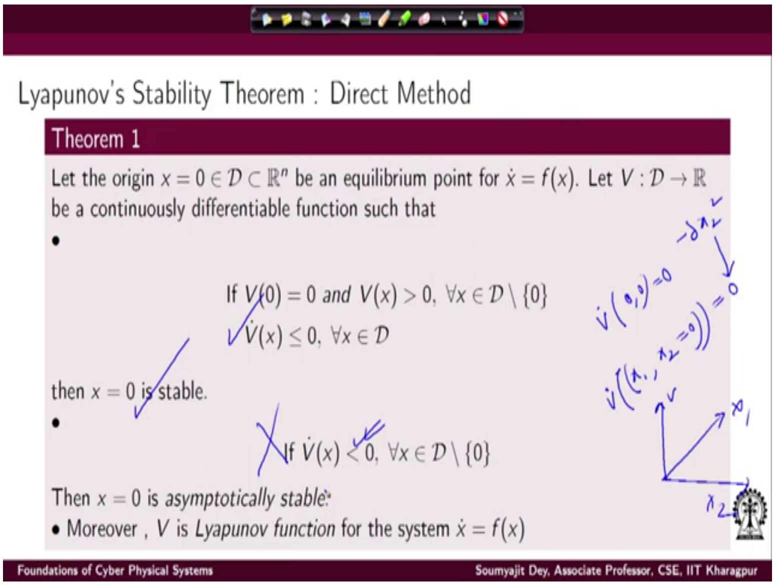

But which kind of stability let us understand we have we have actually shown this right. derivative of x is less equals 0 because ah for all points because as long the moment x becomes 0, I mean the derivative is actually equal to 0. Ok. So, we are not able to say that for everywhere I mean ah so, what we have is, ah If I take the derivative, ah at the origin, ah yeah let us let us analyze that part.

So, for this particular system ah that we talked about if we take the derivative ah at the location. So, if you if you remember ah the derivative was minus delta x_2 square. Right. Ah So, for that if I take it that origin, it is 0. Right. And the condition here is that the derivative is less than or equal to 0 everywhere. So that is why we said it is stable but this was about the standard BIBO (16:26) stability that I can always bound the output region that or given that output region.

I can always bound the input region and say that if I start from this input region, the output will be bounded. What about the asymptotic stability? So, if you remember for asymptotic stability, our condition was stricter. Our condition was that well for every point other than the origin. Well, at origin ah this is happening but what about all the other points? Is it that for all the other points I can show, the derivative is strictly negative.

As you can see here ah I may not be able to show right because this derivative is minus x_2 square. So, ah you see you take any point V for any positive value of x_1 but x_2 equal to 0. This derivative is 0 right so that means it is satisfying this condition. And it is not satisfying this condition because I I have so many points, basically anywhere I go ah in this plane. This is V, this is x_1, this is x_2 I I take any point where, ah I mean which is on x_1. right Ah

There also, this derivative is 0, so, there are points outside the origin. Where also this is is equal to 0. It is not less than 0. Right So that is why we cannot show that it is asymptotically stable using this Lyapunov direct method. Ah. But there are methods like invariance principles, ah through which we can actually prove that but not by this method. Fine.

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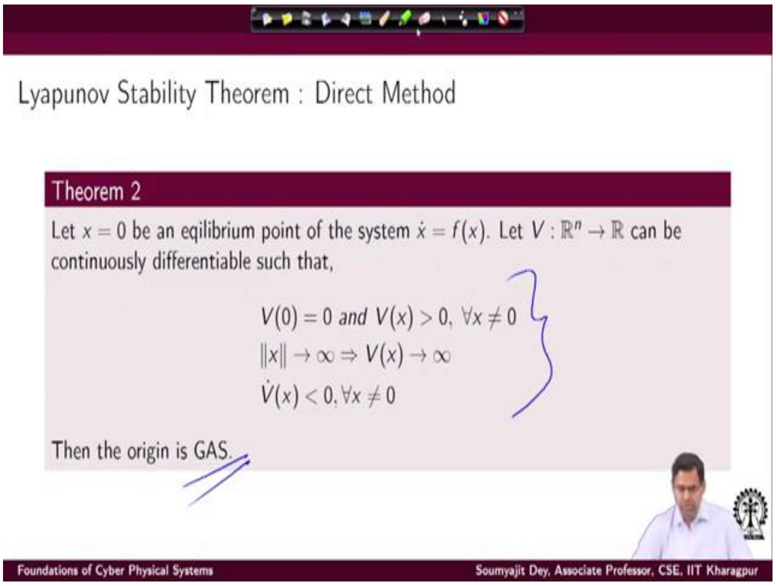

So, fine with this let us now go forward, ah to ah the next conclusion we can draw OK, ah which is the sorry this should be what we call as the indirect method. So, what this theorem says is a bit different from the previous one. So, let us see we we are saying that we will let x be ah an equilibrium point for this autonomous system. And we have this function V which exists and it is continuously differentiable.

And we have this condition that well it is 0 at the origin and it is positive everywhere. And it is strictly negative for all points ah which are not the origin. And V, I mean with x approaching infinity V will also approach infinity. So, under all these conditions, what we can say is that the origin is GAS. That means the origin is Globally Asymptotically Stable. Right. So, let us, let us understand what this really means? Ah Sorry. Ah

This is actually my direct methods theorem only ah but we are actually trying to say that well when this origin is actually globally asymptotically stable. So, what we are saying is that, well as long as these conditions are satisfied ah the origin is globally asymptomatically is stable. Now, let us recall what was global asymptotic stability? It meant that well ah you need not concern yourself to figuring out some region around your origin.

Some region with some well-defined radius delta or something. As long as your system starts anywhere inside it is domain of definition, I am giving you a guarantee that the system will eventually ah eventually approach the origin and in the limiting condition it will kind of reach the origin. Ah. So that, So that is what we mean by this globally asymptotic stability. Right

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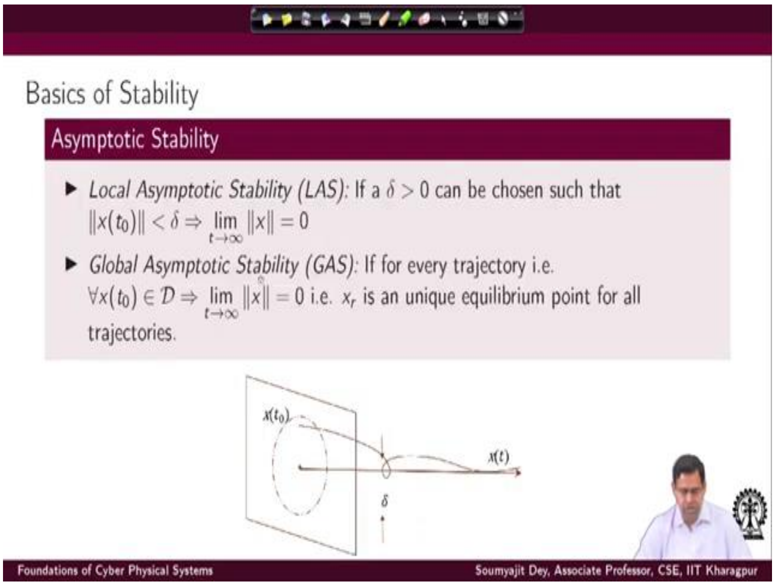

And so, just just for ah a bit recap ah if you want to see our, we already gave you this definition. So, this was our GAS definition that for any point where you start, you will approach the origin eventually.

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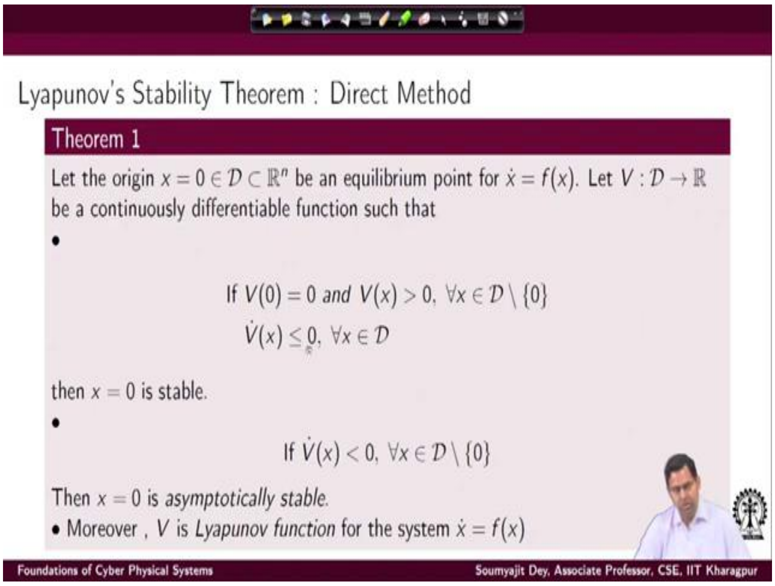

So that was GAS and as part of Lyapunov in indirect method, ah Lyapunov direct method, we initially gave the conditions under which it is locally stable. It is asymptotically stable. Now, so, just note that note the conditions required this is generic and then you had this condition that V dot should be negative or equal to 0 and it will be strictly negative. Then it is asymptotic.

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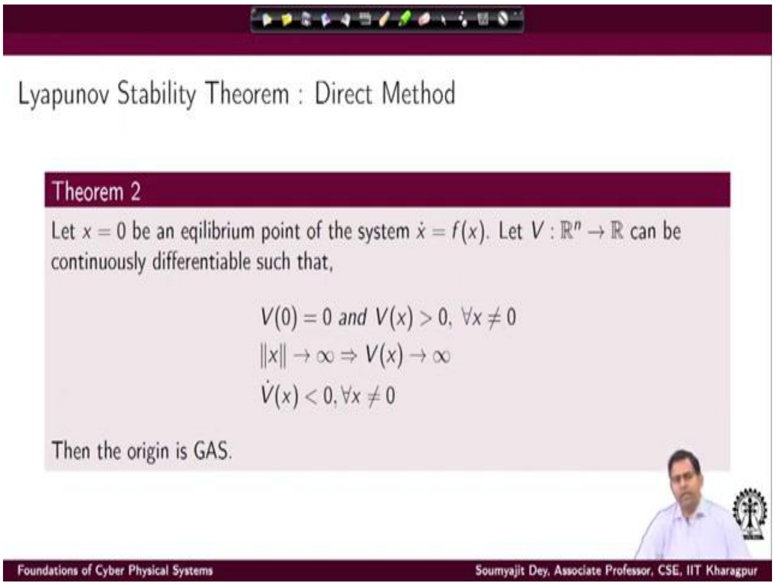

And we are now going for a stricter condition that is globally asymptotically stable. What we are saying is additionally this. We are saying that well ah if your you you have the usual conditions it is 0 at 0. It is always positive elsewhere and it is strictly negative everywhere, apart from the origin. And with x increasing if V is also approaching infinity then we can say that well ah the the origin is the origin is globally asymptotically stable.

That means I can start from anywhere and I will reach the origin. Now, let us let us understand why this should happen? I mean it is easy to easy to argue because if we have this all these properties kind of holding together. Ah, then ah what what is the situation that? As I go far away from the origin ah my value of Lyapunov function is increasing Right, is approaching infinity. And since ah my ah my my time derivative of the Lyapunov function is negative.

That means well with time I must be approaching the origin only which is my position where I will settle down to V(0) equal to 0. Right. So that is why I can start from anywhere ah I need not have an outer bound or something to compute. I took out to compute the inner bound from where I start etcetera. right ah. So, I need not always be concerned about local stability but as long as I have the situation that the moment I go out go away from the origin, my x increases ah V(x) will be ah V(x) will be approaching infinity.

As long as I have that along with all these conditions ah I can say that well ah I can start from anywhere inside the domain of definition of the system. And I will have ah global asymptotic stability. That means the system is stable, starting from anywhere in the domain of the system. Ok. So, ah we can we can have some examples on when this happens for, for What kind of systems we can show that? Ah

The system is GAS using this stability theorem. And we will we will do some examples on this in the tutorial., fine.

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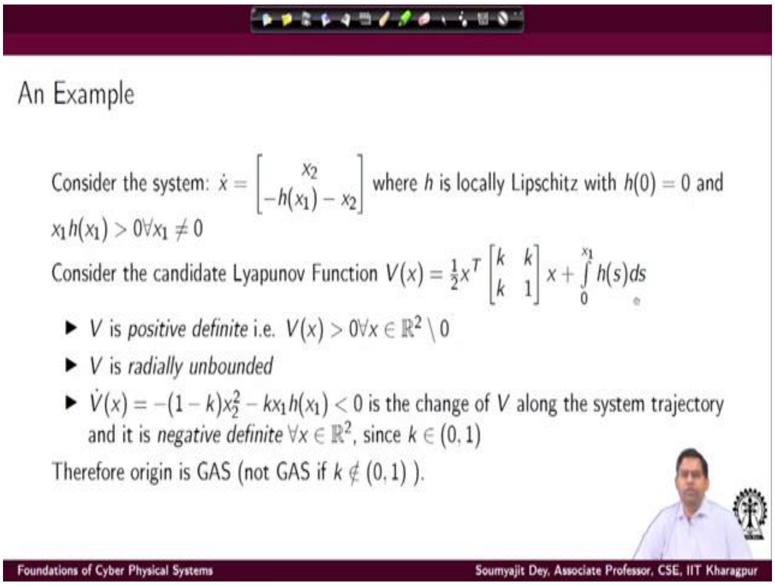

We will have one example here also. So, let us consider this system, so, it is a derived is defined using these two equations. And h is given as a locally Lipschitz continuous function and h is 0 at 0. Ok. And also ah we have some other conditions like ah $x_1h(x_1)$ is greater than 0 for all other points apart from the origin, etcetera. Now, from this system Ah again you see we have some specific choice of Lyapunov function.

And we can show that well this function is positive, definite. And it is radially unbounded ah that means ah with x increasing towards approaching infinity V will also be. right And. So, ah V is positive definite that means for all other points apart from the origin. ah V(x) is positive, strictly positive. And we can show that well the derivative of V if you compute from this equation, I think it would be easy to compute.

Because ah you have to again use the standard template you take the derivative with respect to x and then that derivative we will multiply with x dot. And eventually here you can get this kind of a form. Now, if you get this kind of a form, ah you see Ah you you have this ah I mean let us see that what what are the things I can comment? So, we can say that well ah as long as I have this k to be inside 0 and 0 and 1 I mean it is less than 1. right ah

So, this is a negative term ok, and since the definition here assumes this thing that x_1 ah $h(x_1)$ is greater than 0 and k is also positive. So, this is also a negative term right because overall, with this minus sign this is a negative term. So, overall, this derivative is a negative term. right So, what will happen? This with time the change of V will be I mean towards 0. right Therefore, ah this system is globally asymptotically, stable. OK.

And but if you you see that for other choices of k in this Lyapunov function, suppose k is larger than 1. right Then I cannot say anything if k is other than 1 then these negative so, overall, this positive. So, I cannot make such comments. right But ah in general, as long as I choose, this parameter k to be less than 1 but also positive. And I have this function h defined ah like this that for all a x_1 non-zero ah x times ah h(x) is positive.

If as long as I have this definition, ah we can see that ah this is a function ah which will keep on increasing with x. That means it will satisfy this condition of Lyapunov theorem 2. Right. So because if you see if x increases right ah this is the first term in the function is quadratic in x. Right So, it will always be positive and it will increase in square with x. right. It will increase in square with x right and here in this function. right ah, you are increasing the integral limit. Ok

So, ah you will be integrating h and you will be increasing the value of the limit over which you integrate it. right So, overall, this V(x) function with x increasing Ah it will increase only. So that is why it will satisfy the radial unbound readiness nature. And overall, with all these conditions satisfied, you will meet Ah this requirement of the of the theorem which says that well. ah If these things happen together then the system is GAS, fine.

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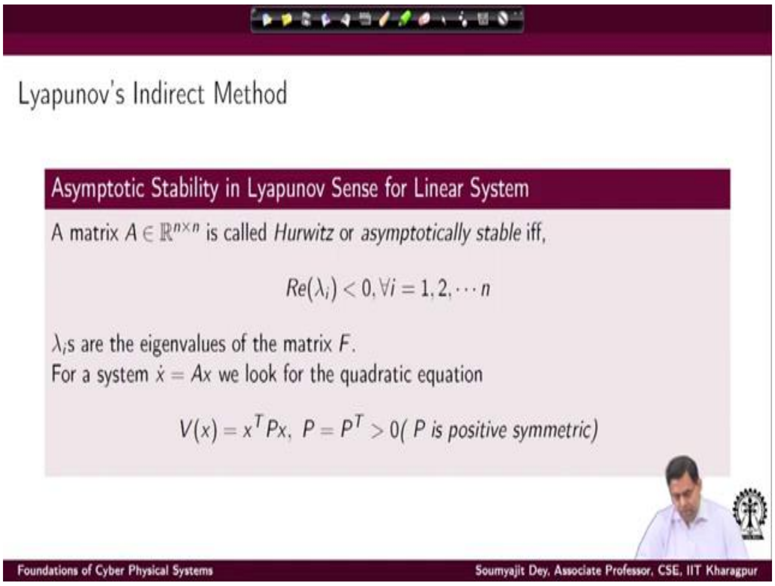

Now, ah the next ah statement that we have is Lyapunov class of indirect methods which primarily mean that well we will now try to see that well, how this Lyapunov style argument will go for linear systems? How this Lyapunov style of argument will go for non-linearized versions of non-linear systems, etcetera? And accordingly, there will be this Lyapunov indirect methods. So, fine, we will take this topic, ah in the next lecture again and further time. Goodbye.