Foundations of Cyber Physical Systems
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Lecture – 38
Lyapunov Stability, Barrier Functions (Continued)

Welcome back to this lecture series on Foundations of Cyber Physical Systems. So, in the last lecture we have been talking about Lyapunov Stability and the Stability Theorem. And from there let us go ahead with the proof of this theorem.

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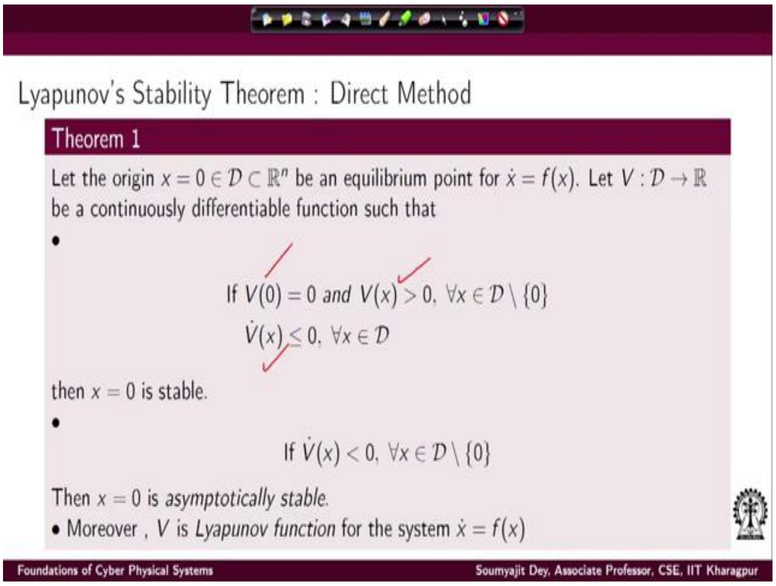

So, if you see this was our claim here that as long as I can figure out, the Lyapunov function which satisfies this condition, this condition and this condition that it is positive with time always right and it is a is 0 at the origin, so yeah is 0 at the origin. Ah it is It is positive everywhere else and it is gradient is negative right ah as long as that is true then we can say that x equal to 0, the origin is stable and if further this is strictly less than 0. And then we can say that this is asymptotically stable.

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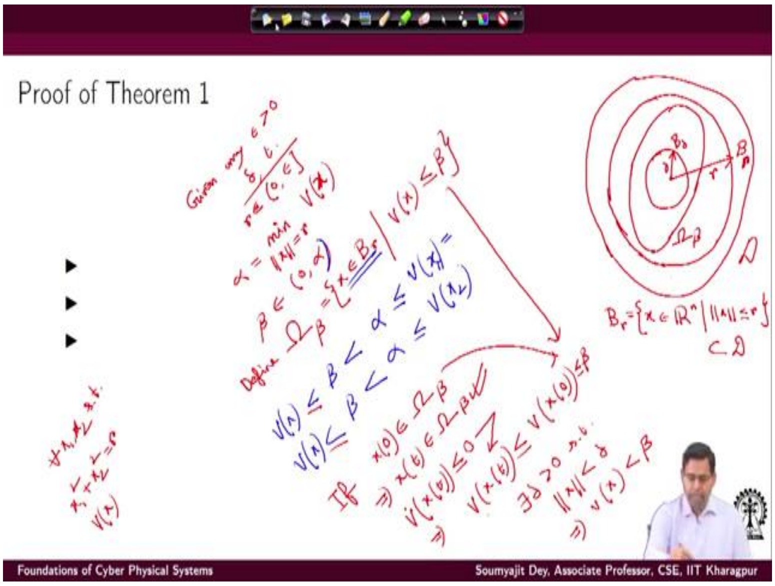

So, kind of let us go about proving this. So, if you remember our stability requirement. In the first proof, we need to show that the system is stable, right. That means we are given the outer ball of radius ϵ. And we need to show that there exist a time point t naught beyond which the system must be going inside and in your ball of radius δ. So, this essentially means that given this ϵ, I should be able to compute this δ and time, right.

After which it is inside δ. So, as long as I can show that then I can say that well, this is stable. right So, suppose I am given this ϵ. My objective is to compute this δ. So, what we do is first, we will choose a smaller value, r which is basically something smaller than epsilon but greater than 0, of course. right And with this, so, let us say this is my domain of definition, this is my D. Okay.

This is the origin and I have this sphere of radius, r some B_r ok. Let us say this is radius r. okay And then so, we can say that B_r is I mean we are formally defining this. It contains all those points in R to the power n such that the norm of x is less than or equal to r. So that is the definition of the B_r, right and it is of course contained well, within this domain of the system. Ok. Now, what we will do is, we are assuming that such a Lyapunov function V exists right with all those desired properties of being 0 at the origin, negative or equal to 0 in it is time derivative. And being positive everywhere. right So, let us evaluate this function everywhere on this circle. ok So, all I am doing is saying is that well, these are all I mean the positions on the circumference. Ok. On all these positions ah we can evaluate this function V(x) because it is defined everywhere here and take the minimum value. Ok.

So, let this be given by some alpha which is minimum. So, you are taking this radius. And you are taking all points which lie on this radius and you are evaluating V(x) on them and that minimal point is what we call as α, right. Now, you choose a value β which is between 0 and α. Ok. And then let us define another inner set of this sphere B_r. Ok. So, you are defining like this. These are the points.

These are those points of B_r for which V(x) is less or equal to beta. Now, let us understand why we say that this is definable, because what we have done, We know that well, what are the points which are lying on this red on, on this on the circumference of B_r, right, because for all of them the system norm is equal to r right. So, it is a 2-dimensional system, let us say, in then we are talking about all x_1, x_2 such that $x_1^2 + x_2^2 = r$. Right.

And for all such points, we are saying that well lets let us evaluate V(x) for all those points and see that well what is the value of V(x)? right And then you you will get that value. Right. So, ah this this can be computed and once this is computed, ah you you just take an another you just take any any value between 0 and α. Now, once you have done that we know that well these are the points where V(x) is, is being computed and the minimum value among these points, where V(x) is computed that value is α.

And now, we are we are saying that we are trying to define this set β this Ω, this set Ω_β, saying that well, we are considering all those points of B_r, this sphere of radius r, for which the value of V(x) is less than β. Ok. Now, let us say this gives me a region like this, so, this is my Ω_β. Now, the question is well, ah why should Ω_β be inside r? How do I have this guarantee of then? I should be able to draw this picture.

The guarantee is because let us say ah I do not have this to be completely inside B_r. It is also containing some points which is beyond this ah this circle B_r. That means that also means ah in that case, ah Ω_β will be containing points on B_r. Then Ω_β will be containing points where

V(x) will evaluate to α. right ah Let us say let us just assume let me just draw an alternate picture. Let us say this is not Ω_β but rather something like this is Ω_β.

Then what will happen? I have a point here, right I have a point here and I have a point here which are all members of Ω_β and they are also located on B_r. That means at this point let them be x_1 and let them be x_2. And then for these points Ah $V(x_1)$ um will be equal to some value ah which is either greater than or equal to α. right And similarly $V(x_2)$ is also greater than or equal to α, as per this definition of as per this definition I have here. Right.

And that I have chosen this value beta to be such that it is between 0 and α. And what is α? Alpha is the minimum of all the points on which ah alpha is the minimum value V(x) can assume on this circumference. So, let us understand this very clearly. I took I am just repeating, so, I took this sphere and I evaluated V(x) on it is surface and that gave me possible sets of values. Right. And among those values whatever is the smallest value. OK.

The place of where the distribution will dip. I am taking that as my ah that is that that as my ah value alpha. right And then I am choosing a parameter β which is smaller than that and I am defining a space Ω_β. So that is defined to be a space which is also a member of B_r ok, and for it V(x) is less than β. Now, if I say that well, I am doing going like this I have a issue. The issue is that well, ah first of all I am having points ah where I am having points ah where I am not inside B_r.

So, I am not satisfying this and also suppose I mean how do I know that I am inside it. I am not crossing over or I am touching it because if I touch it then I will have points x_1 and x_2. Right. And at those points, the value of V would be greater than or equal to α. But we know by construction Ah that we I mean all the points which are inside or on the surface of Ω_β there ah the value must be less than α.

The value must be, because beta is less than α because this is an open interval here. Right And the value of V(x) will be less than β. right So that is how I have defined it. So that means well. This is ah this is a, this is a closed volume Ω_β which must be located strictly inside this circle that is B_r. right So, this is inside B_r, the omega beta is actually inside B_r. Ok. Because by our definition, we are looking for all those points which are less for, for which V(x) is less than β and in case Ω_β was touching B_r.

Then I would have points which ah for which the value is greater than or equal to α but by definition, Ω_β is only going to contain points where the value of V(x) should be less than or equal to β. And β is strictly less than α so, this thing will not be satisfied. I hope this is clear. right So, this only says that well, Ω this Ω_β must be inside B_r. ok I mean it should not event which it is kind of strictly inside B_r.

Now, let us observe one thing, suppose you pick up any location x which is inside Ω_β. And you say that well, the system at E equal to 0 is located inside Ω_β. ok Now, we should be able to show that if that is so then at any time in future, it will be located in Ω_β only, question is why? Because you see the derivative of V over x which is a function of t is negative. Right.

That means the system can be anywhere but V, when I evaluate V on the systems trajectory at some future point, it must be less than or equal to the value of V at the system ah system location at t equal to 0 right, because with time V is decreasing. So, the value of V must only decrease. right And we have assumed by construction that x(0), is inside Ω_β. Now, this definition says that well, if x(0) is inside Ω_β then V(x(0)) is less equals β.

Now, by the definition of the Lyapunov function we have V(x(t)) as less than V(x(0)), so, we have V(x(t)) is less than β. Now, if that happens, the only way for that to be true is that x(t) is inside omega beta. right so So that means this is like an invariant set and it is able to bind the systems trajectory. And it can say that well, as long as the system starts inside Ω_β, the system must be inside Ω_β for future forever. Right.

Because other otherwise the Lyapunov function's guarantee will not hold because the functions design is such that it is supposed to dissipate. And it is a it will only become small and by construction all smaller values must be inside this set Ω_β. Right. Now, further, we can say that there exists delta, greater than 0. So now, let us kind of draw a smaller circle here. Radius δ now, this is your B_δ. And since this guarantee holds for the entire set Ω_β, I can write this also, sorry.

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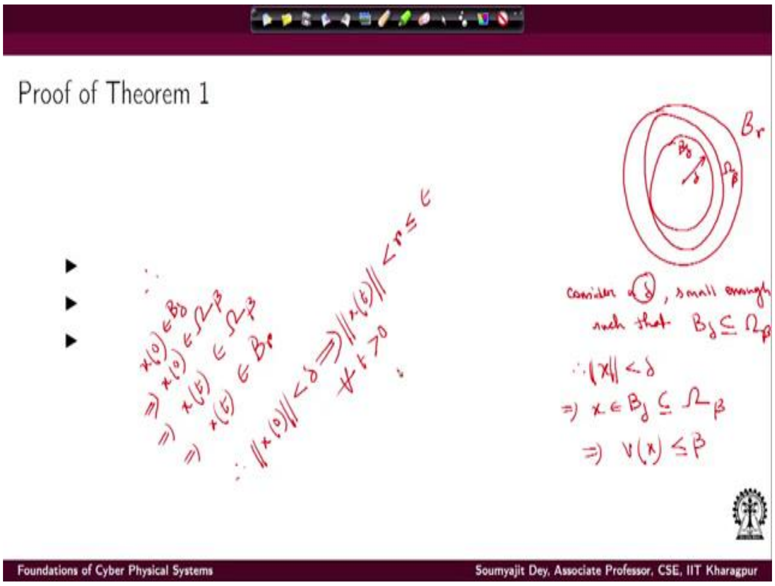

So, just because we can write that if the system starts inside Ω_β, the system will remain Ω_β. I can and if we take a radius of a circle, such that, of a sphere which is small enough, so, I choose δ in such a way that the resulting sphere B_δ is completely inside Ω_β. Then what can I say? I should be able to say that as long as systems norm ah is less than δ right x is norm is less than δ. Ah

By definition what I mean as long as for all x which are inside this circle this sphere B_δ. By definition of this sphere system x norm is the I mean basically I mean x is located inside. right So, the system is not the norm is less than δ. Right. So that means, ah since this is less than delta. um I can say that well, I mean, since B_δ is completely located inside Ω_β ah let me just rewrite this again.

Now, since this B_δ is located inside Ω_β, I should be able to say this right like as long as I pick a location x with this property, it implies that will x is a member of B_δ. right So that means x is also a member of Ω_β. And then by definition of Ω_β what I have is, V(x) is less than or equal to this beta. Right. So that is it right. I mean the point is the moment, I am able to create this Ω_β.

I am able to create this inner circle B_δ. That means the point is I can create this δ. Ok. If x starts inside this, then x will be a member of I mean it will, then that means x is also starting inside Ω_β and that means in future xt is located inside Ω_β. That means in future x is x is bounded to be inside this outer circle B_r. Ok. So, as long as I am starting inside this in future, I am giving a guarantee that I will be located inside this.

Which is less than equal to epsilon for all t greater than 0. Ok. So that is about it, like we can say that I can show that this bounded region exists inside which I must be. Ok. Now, we can also show the asymptotic stability essentially, it means that it turns out that it is sufficient, to show that V(x(t)) I mean it will tend to 0 as t approaches, infinity. And since V is monotonically decreasing the I mean x will be approaching the origin there. Right.

So that is about it that is about the proof of theorem 1 and with this we will end this lecture. Thanks for your attention.