Foundations of Cyber Physical Systems
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Lecture – 37
Lyapunov Stability, Barrier Functions

Welcome back to this lecture series on Foundations of Cyber Physical Systems. So, we will start with our week 8 coverage.

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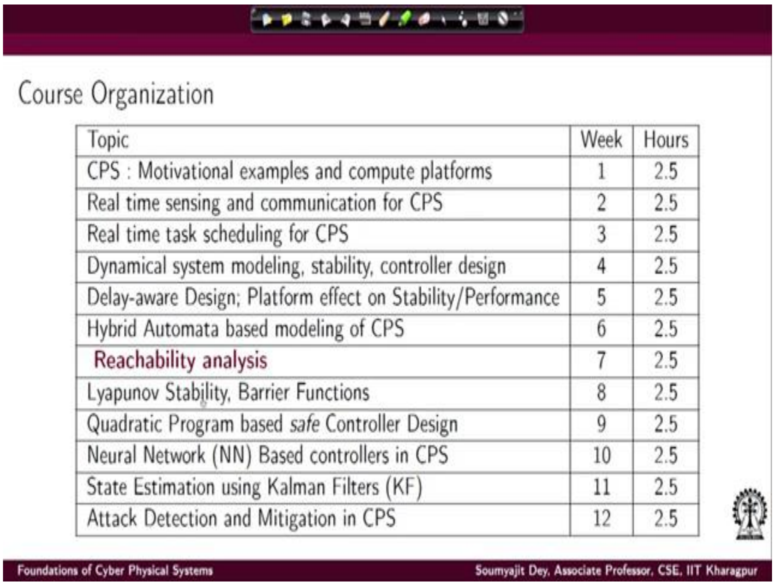

As part of week 8 topics we will be talking about ah primarily, this Lyapunov stability and barrier functions. So, just to give you an idea what this really means.

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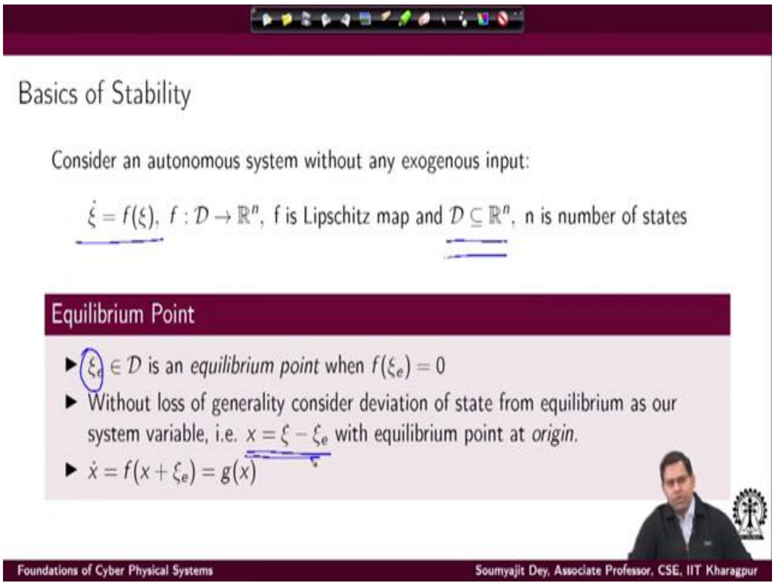

So, we have understood a notion of stability ah that well if there is some disturbance in the system. The system must be able to reject the disturbance and come back to the to the to a stable equilibrium which is basically the region around it is set point, let us say. Right. Now, the techniques of stability analysis that we have discussed till now ah are more like applicable for linear systems. But ah let us also understand that there are quite a few examples of control systems which are non-linear.

Where you can, of course, analyze them by linearizing them around some operating point. But there also exists some general techniques for ah discussing on stability and analyzing stability. For General autonomous systems, where we do not make such assumptions that whether the system is linear or non-linear things like that. So, when we will be discussing stability now, the difference is, we want be making any assumption about the structure of the system.

But rather we will be saying that well, it can be any autonomous system without any exogenous input. That means ah the way I mean we are basically considering a closed loop here. So that the dynamics is a function of the current state of the system itself. ok. And we will say that well ah the only assumption that we will make about the system is that it is some function ah on an n dimensional state space and it is a Lipschitz continuous maps.

It is a Lipschitz continuous map, and so and it is defined ah on some domain of operation which is basically of subset of this R^n, that is the n dimensional state space. Ok. Now, let us say that we are trying to stabilize the system around some equilibrium point ξ_e. OK And so, the way we will be ah discussing about well. What really is an equilibrium point is that? We will consider that the deviation of state from the equilibrium.

Let that be denoted by our ah system variable x OK. And so and we will assume in general that the equilibrium point is located at the origin. What I mean is that well, the equilibrium point that means this point around which the system is trying to be stable it can be anything. But what we will do is? We will assume that the definition of state where we will assume origin shift. So that we can say that ah the system is trying to stabilize around x equal to 0, like that.

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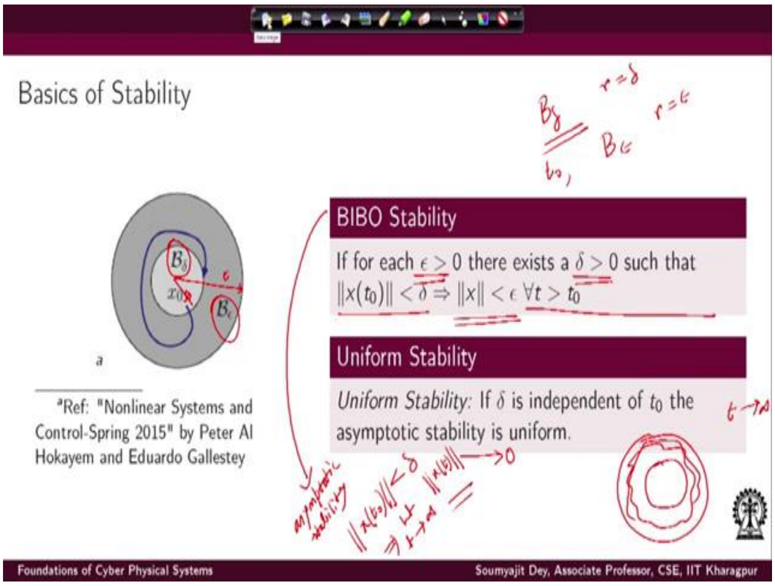

So, we will begin with this basic notion of stability of systems and we will talk about bounded input, bounded output or by BIBO stability and asymptotic stability of systems. So, let us understand what this means. So, in this on the left-hand side we have showing you a picture ah where we have an n dimensional sphere, B_ϵ. So, this is a radius epsilon and from this origin x naught we have a smaller inner sphere B_δ Ah with let us say, the radius is delta.

So, what it means by BIBO stability is that, for any such outer sphere, definition that is for each such epsilon, there exists this smaller sphere with the parameter delta, such that if the system starts inside the inner sphere that means the norm of $x(t_0)$ is to be less than δ, then after some time point t_0, I have a guarantee that the system must be somewhere inside this outer sphere ah B_ϵ.

So that means we are able to say that well the output is bounded with the input being bounded. Ok. So, let me just repeat we are saying that well we are trying to argue that the system will settle down at some position. Even if the system does not settle down exactly at a position, in this case, like we discussed earlier that we are doing a shifting of the origin. And we are assuming that well the system is always going to stabilize around, I mean we are shifting the origin to the set point and we are arguing around this origin as the position of equilibrium. Ok. So, what we are claiming is simple. We are claiming that well we should be able to identify this region, inside which I mean this region around this equilibrium point where the system will eventually stabilize. In simple terms we say that well for any such bounded region, where I want to guarantee the system will be able to I mean the system will be residing.

I can also I can always compute a bounded inner, a bounded region where the system will start. So, we are saying that well for any such bounded region bounded eventual region, let us call it eventual because that is where it will be located ah in future. Right. For any such bounded eventual region, I can define a bounded initial region. And I can guarantee, with the with also a value of time point t_0.

That I can guarantee that as long as the system starts from this bounded initial region B_δ with a radius δ, as long as the system starts from inside this initial region, after this time point t_0 which also I can give the system is guaranteed to reside inside this region B_ϵ with radius equal to ϵ. So that is the guarantee I am giving. I am saying that well let the system start inside this bounded region.

The system must be inside this outer bounded outer bounded outer region or in other words, I am saying that well ah as long as you I you want me to guarantee that the system must be located inside a region, I can give you a limit on where the system must be initially located and I can guarantee you that as long as you start the system from that initial region after some time point which also I can compute, I can tell you that the system must be located in the inside the target region.

So that means once you bound the output, I can bound the input from where the system must start so that eventually the system ah is steered towards that target region even target region which is your bounded output. Ok. So that is a simple concept of BIBO stability. All it means is that as the moment, you define this B_ϵ region, we should be able to, we should be able to talk about this B_δ region. Ok. I mean. ah I mean we should be able to define B_δ .

And we should be able to give you an initial time after which the system I mean as long as the system starts from B_δ , after that initial time, the system is guaranteed to be inside this B_ϵ. ok ah. So that is the point. Now, the other thing is ah, we can also say that in not only ah talk about we can we can talk about bounded input and bounded output stability. We can also talk about asymptotic stability. And what it what it says is something additional.

Here we said that well I am giving you an input bound which guarantees this output bound. In asymptotic stability I will say that well I can give you an input bound that your system must start inside this bound. And as long as it is turn starts inside this bound in the limiting condition that means in future eventually, the system will approach the origin which is it is point of equilibrium.

So, when I say bounded input bounded output, I say that well you give me a bounded region where you want the system to be located, I can give you the corresponding bounded initial region. Now, this location of circles, forget about them I mean whether they are concentric or like that. The important point is I am, I mean who is located inside whom that is not the point. The point is as long as you give me a bounded final region, I can give you a bounded initial region and I can tell you that as long your system starts from inside from inside from inside this initial region. I give you a guarantee that eventually, the system will be inside that bounded region, where you that you give me that is B_ϵ. For that B_ϵ that you give I can always compute for you this B_δ and you have a guarantee that as long as the system starts from B_δ , eventually, after a well defined time point, the system will be inside B_ϵ that will it will keep on being inside B_ϵ. And what this asymptotic stability tells me? That this B_ϵ will not be a region and I mean well it can we initially a region but with time tending to infinity. So, here we do not have any definition. When we talk about BIBO stability, we have been we do not have any well-defined notion of where the system will go Ah with time approaching infimum.

It just says that well the system can be anywhere here. Right. The system can be anywhere here inside this epsilon ball. Right. But in asymptotic stability it says that well with time in future, ah with time tending to infinity this epsilon ball will collapse. It will collapse to the origin. That means, ah I am giving you a higher guarantee I am not telling you that you start from this initial region that I designed for you.

And it is guaranteed that you will be inside this bounded region B_ϵ with which we started the analysis. But rather I am saying that this ϵ region that we that with which we started to analysis that will eventually collapse, the sphere will become smaller, and eventually your system must reach the origin. It is not it will it need not be lurking around inside B_ϵ but it will eventually reach the origin. So that is asymptotic stability.

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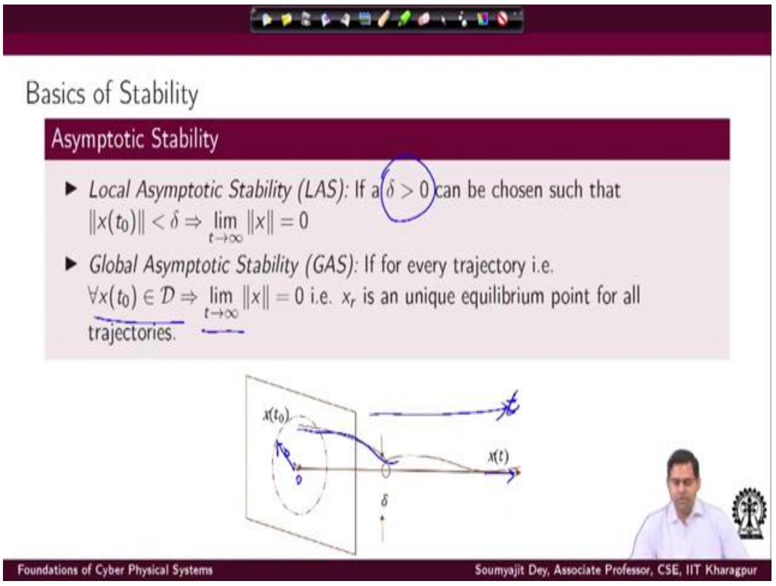

And then we have this concept of local asymptotic stability and global asymptotic stability. So, we have been able to define this notion of bounded stability and then from this, let us go to this definition of asymptotic stability. And before that you let us also remember that we have also said that when is the system uniformly stable as long as this computation of this δ is independent of t_0.

That means we can always say that well ah as long as I am always able to say that ah well this δ exists, and the computation of this δ is not a function of t_0. That means that in we I mean after t_0, it is inside this delta and there is no dependency between them. Then it is a uniform stability. Now, let us talk about another, more advanced notion that is asymptotic stability.

So, the first definition is about local asymptotic stability or locally asymptotically stable in different books you may see these different terminologies. So, what it says is that well if this delta can be chosen such that ah we can say that in the limiting condition, ah this x will approach the origin. Then I would say that while the system is LAS or Locally Asymptotically Stable, ah at the origin. Ok. So, as long as we can say that well at this time t_0 Ah the system is inside delta.

That means we are saying ah let us look at this picture now x is location, ah at t naught. Ah. It is inside δ and if it is existence inside this inner wall. Also, additionally guarantees that with progress of time I would be approaching or that means the system would be approaching the origin. Ok. So, this is the origin and this is showing the progression of time here. right ah. So, as long as with this time, I am approaching ah this ah origin here.

So, I would say that it is locally asymptotically stable. And if we can say further that for every trajectory that means it is not only valid for ah this small region around the origin. But this notion is valid for anywhere in which the system is defined. That means the domain of definition D OK, for all locations, wherever I start. ah If I can tell that well eventually, in the limiting condition. ok The system is going to reach origin. Ah

Then I can say ah that well it is Globally Asymptotically Stable or GAS. Ok. So, let us just understand the difference when we say look at locally stable, that means I am I am able to say that well the system is stable around the origin or in inside a well-defined region. I am, I am able to choose this region with a well-defined radius. As long as I am able to guarantee that there exists this small region. So that somehow, if you can steer the system inside this small region.

You can steer the system inside this small region and you are able to compute this delta. And you can guarantee that well somehow, if the system starts in this inside this region then it will approach the origin it is local stability. And definitely it is global means I can argue like this, without even finding the delta. So, in local I can give the guarantee only inside this sphere of radius δ.

But in global icon I am just able to skip this guarantee for anywhere, ah for any for any trajectory starting anywhere in in the domain of definition of the system. ok So, in that case I can just say that well it is a the system is GAS or Globally Asymptotically Stable.

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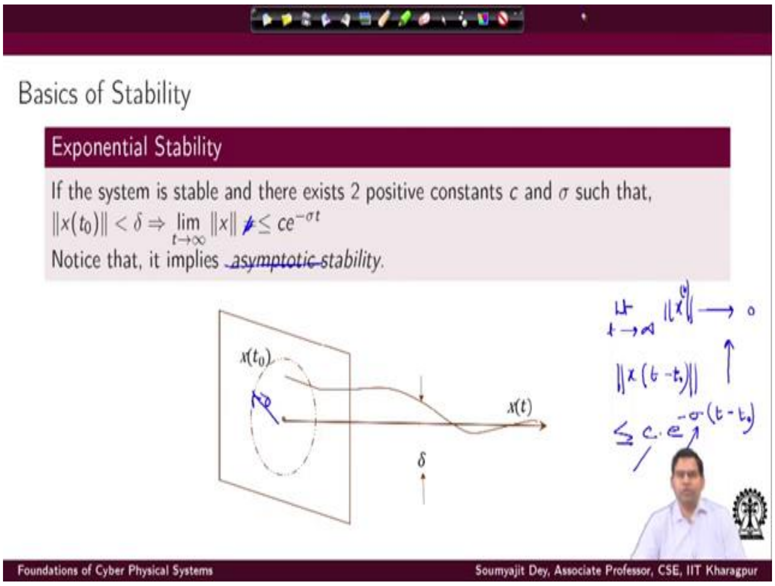

So, ah in I mean just like we are able to talk about asymptotic stability, we also have this notion of exponential stability which essentially means that the rate at which the system will dissipate energy and try to become stable will be exponential. That means, if the system is stable and I can figure out two constants of an exponential function. Let us say they are c and σ. And we can say that well ah so, as you can see that this is also a local stability condition. Ah

That will as long as I am inside this delta ah then with time, I am able to guarantee that I am going to approach the origin. Ah, but at the same time I mean my value ah is following this decay. Ah Then that means I am implying an exponential stability here. So, this is exponential, sorry about it. Ok. Now, lets let us understand what it means. So, if you see with time the system's ah norm will again be approaching the origin. Right. But at every time point t, let us say t_0 is the initial time.

So, $t - t_0$ let us say the value of this norm. right. It will be bounded by this exponential decay function with this parameters $t - t_0$ here, or that the system's norm with time I mean I can always bound it by this exponential decay function. For which I am able to compute this parameter c and σ. Ok. So, ah as long as I am able to do this, I can just say that well this system is not only asymptotically stable but it also exponentially stable.

If you see that exponential stability will imply the asymptotic stability. Because if this happens, this is actually implying asymptotic stability here, of course x is a function of time. Okay. So, let me just repeat again. We are saying that well if the system starts inside this sphere of radius delta at time t_0, with time the system will approach the origin. But not only that I am able to give a rate of the approach or better to say, I am able to upper bound a rate of the approach.

That means I am saying that I can actually figure out these parameters of some exponential decay curve and I can say that well with time this system's norm is ah, or I mean here in Euclidean space we are actually go I mean we are actually by norm we mean this radius value or the distance from the origin. Right, ah so, ah that is kind of upper bounded by this function. ok So, since this actually implies the limit also. So, exponential stability is like a stricter form of stability.

So, you can actually get systems for which you can prove asymptotic stability but only a subclass of them would really be exponentially stable. So, ah just to give you a context, why have we suddenly starting to discuss all this is because ah these concepts of stability and the theorems that we will talk about ah are actually used to design ah modern control algorithms which apply to a large class of systems ah for which ah the simpler control algorithms that we discussed may not be applicable for both designing their controllers and also for analyzing their stability. Those algorithms may not be applicable.

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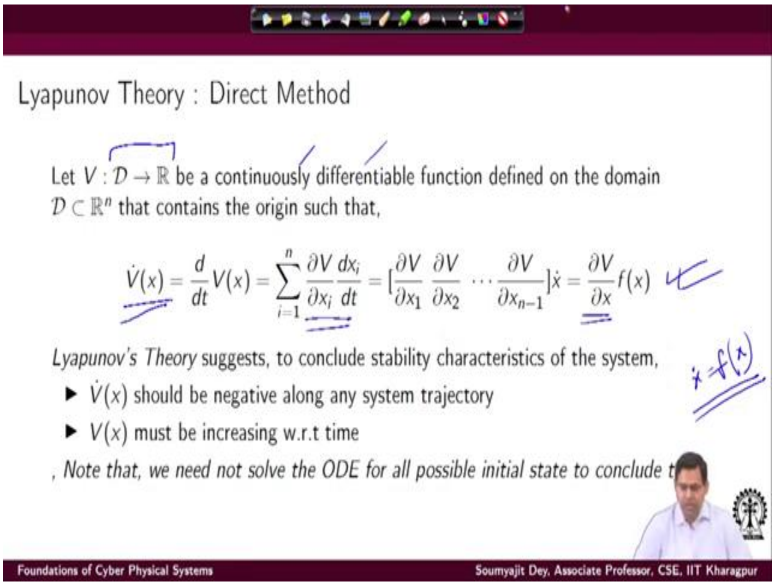

So, fine, in this regard, we will be introducing you to this famous theory of Lyapunov. Ok, So, this is a method to ah kind of ah analyze a systems, a general autonomous system, stability. So, historically this has I mean this has been a proven method. Right. So, when this techniques was were discovered by this Russian mathematician well they found in future they were later widely used in several kinds of artifacts design used in world wars and other places.

So, slowly this became a part of modern control theory and Lyapunov theory actually has got widespread application across control or general optimization and even AI techniques nowadays. Ok. So, ah let us first talk about ah the direct method of Lyapunov. So, essentially, this is a method to argue about stability of the system. And what this method says is that well, for a given any system F ah Let us say I mean so, you are given this system Ah $\dot{x} = f(x)$.

We are trying to talk about, so, this is an autonomous system and we are trying to talk about the stability of the system. We would say that the stability analysis of this term is dependent on the existence of a function V which is called the Lyapunov function. Ok. Now, note that this is a scalar function. So, this is the real valued scalar function which will need to satisfy a property that it must be continuously differentiable. OK.

And it is defined on the domain of the system and it is also defined at the origin. And since it is continuously differentiable, we can write, it is derivative with respect to time as $V(\dot{x})$. OK.

And since this is a multi-dimensional system, it will actually be Ah you have to compute this thing like this. Right. You have to do del V del x_i in every ah system state component. So, it is an n dimensional system. So, let there be n components right and then dx_i dt. Right.

So, you have to compute all the components. And finally, you get this quantity del V del x. right So, overall, $\dot{V}(x)$ can then be this derivative with respect to state ah multiplied by $\dot{x}$. Now, we already know what is x dot, that is the systems definition itself right $\dot{x}$ is f x. So, essentially, ah derivative of with respect to time of the Lyapunov function is basically derivative with respect to space of the Lyapunov of function, multiplied by the system function.

Now, suddenly we are we have started saying that well, I want to analyze the stability of f. And we are certainly bringing in the concept of another function V. How are they related? The relation is by this only we can see that the this is the scalar function defined on the state space and we are saying that well the derivative of this function, the time derivative of this function, ah actually is what links up with the definition of the system.

As we can see because the time derivative is equal to the space derivative of the function, multiplied by the systems function. Ok. Now, what this Lyapunov direct method says is? That in order to conclude the stability of the system, ah we need to prove two things that this derivative of V is negative along any systems trajectory. That means inside the entire state space ah this derivative is negative. But V(x) is a continuously increasing function with respect to time. Ok.

Now, what is important is I do not need to solve this ordinary differential equation for all possible initial states of the system. Now that is the beauty. All we need to do is I need to compute the derivative, the time derivative of this function. And we must be able to say that well, this is the nature of the function. And the from the nature of the function only we can conclude that well the system is stable. It looks like magic but it is actually provable.

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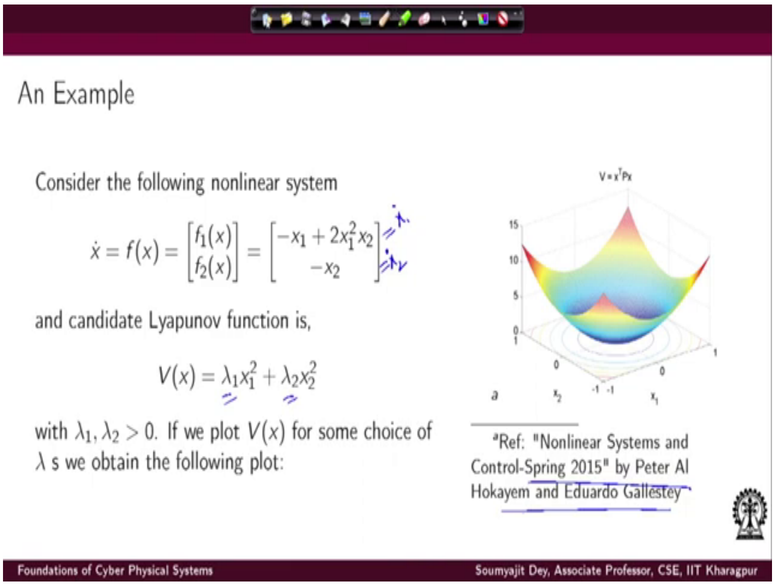

So, let us take an example here. So, let us take this following non-linear system, so, ah $\dot{x}$ = f(x) it is a 2-dimensional system. Ok. So, in different dimensions ah the components are f_1, so, ah f_1 and f_2. ok ah So, it is given by this, as you can see so ah that this only means that this is equal to $\dot{x}_1$ and this is equal to $\dot{x}_2$ here. Right. And we have a candidate Lyapunov function here. So now that is the important thing. We need to understand that the Lyapunov function is never known. Right

So, you have to assume a suitable candidate Lyapunov function. You have to guess that function by looking at this system definition. Ok. And the and there is a trick you have to identify some the existence of some Lyapunov function. And once you can show that this Lyapunov function satisfies these properties. You can actually conclude that the system is stable. So, for this the candidate Lyapunov function is V(x) with parameters λ_1 and λ_2 greater than 0.

Now, for this ah if you plot this function for some choices of this lambda, ah you get this following ah for following plot and this has been copied from this text only. Ok. And what you have here is ah for different choices of this lambda, you will you will be getting this different level sets of the function here. right. And so, with this all these level sets the overall functional form here will look like this, as you can see. OK. ah so the. So, fine Let us let us proceed with this. Yeah.

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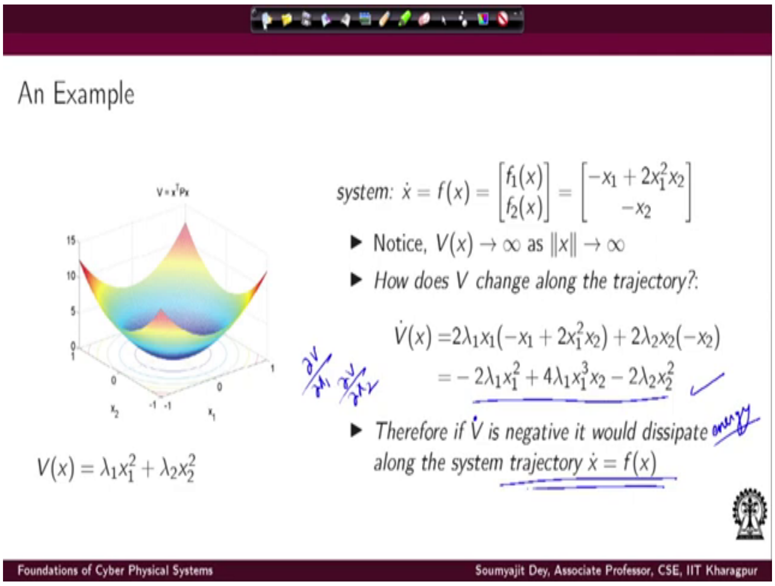

So now, ah to argue further see this is my system definition. Right. And if you notice this functions characteristic, what happens? If you see that as the value of V(x) ah becomes large right, I mean this value of V(x) becomes large, with x becoming large. OK ah. So, ah we had that as an important property right I mean xV(x) must be increasing. Now, ah not only that we need to identify well what is, ah How does v change with time? right Now, ah so, if you take the time derivative of this function. Ah

You can see that for this function, if you take the time derivative, so, you have to basically compute these terms del V del x_1 and the term del V del x_2. Right. And then you have this marketing multiplied by the f_1 and f_2s. And overall ah this derivative will be this and you have to show that under what conditions this function is negative. Right. So, if I can show that this derivative is negative, it really means that the system will dissipate along the trajectory of the system Ah I mean this system will dissipate energy along the trajectory of the system.

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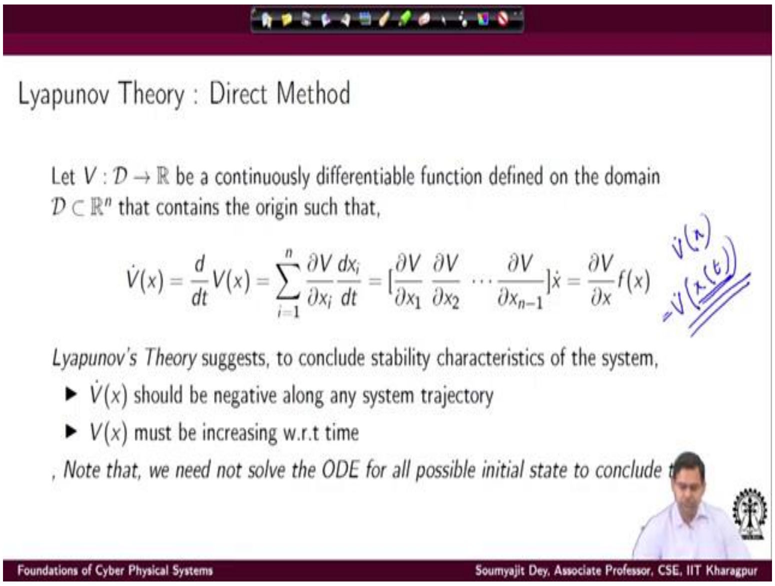

But these are all claims that we are making. Right. We are saying that well if this functions derivative is negative along the system trajectory. By the way what does it mean by the system? V $\dot{x}$ be negative along the system trajectory. Let us understand that where am I evaluating the function? Whenever I am taking this time derivative, ah it means I am doing del V del x times the derivative of the I mean state itself which is the with which is the system function. Right

So, with time where is the system located, of course on it is trajectory. That is why, whenever we are computing, the time derivative and the positions on which this derivative will be defined will be on the trajectory of the system itself. Right. ah Because let us understand this when, when we say V $\dot{x}$, essentially is we are taking time derivative of V which is defined over x and x is a function of time. Right

So that is the relation with time that is coming and x(t) is the system trajectory itself. Right. So, I am actually taking the derivative right on the trajectory of the system itself. So, ah let us let us prove this we have made enough time amount of claims, so, let us make a proof out of this. So, this is Lyapunov first theorem. So, there had been a theory a series of theorems ah called Lyapunov of stability theorems and this is called the direct method and we will see the other methods here. Ok.

Now, ah by the way we have kind of taken this simple example and we have tried to just tell you that what can be a candidate Lyapunov function and if you plot this function with changing parameters, how the function will be looking like. Right. So, here you have x_1 here you have x_2 and in this axis you are basically plotting the values of V. right So, for different values of x_1 and x_2 as you can see that ah this is like a quadratic function. Right.

Since this is the quadratic function, we have written V in a quadratic form here. right ah And this is typically how ah this quadratic function's shape ah would look like? Right And if you project this function, ah into the x_1 and x_2 plane, right, if you project this function into this x_1 and x_2 plane, you are actually having this concentric circles which are what we call as level sets of the function. So that is it. ah So, if you if you I mean we will speak more about level sets later on in general.

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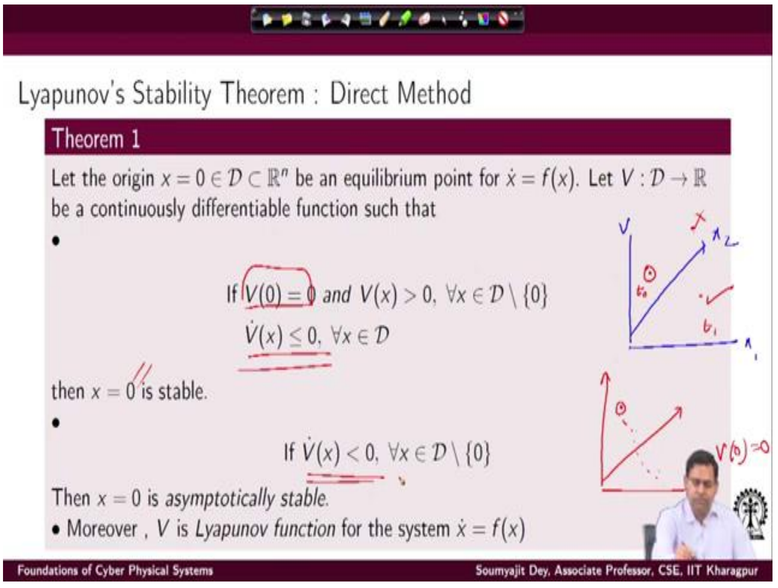

Now, ah let us let us see what see what this theorem is all about, and how we can actually show. That well, if the so if some function satisfy certain properties, it is really I mean I can say that the function defining an autonomous system is stable. So, ah let this be an origin of the system OK and x equal to 0 and and let this be an equilibrium point for the system and the system equation is $\dot{x} = f(x)$. OK And we assume that for this ah this continuously differentiable function V exists with the desired properties.

And there, as per the claims that we made earlier. OK. That V dot is negative and V(x) is increasing. Ah So, here what we really claim is, ah V is ah V is 0 at the origin and ah V(x) ah is greater than 0 for all because V is defined over x right. So, at x equal to 0 it is 0. And for all other positions x that a not equal to 0, V(x) is positive. But for all positions V is this derivative is negative. Ok. So, if we can we can we can claim this then ah the theorem says that well if you can if you can actually show that such a function exists, then the system f ah has got stable equilibrium at x equal to 0 and if the derivative is negative. Then ah so that is the claim and there is a further claim, ah that well in this case we said that well ah v V(0) is 0 but suppose that is not true. And all we know about this function is that well ah it is positive everywhere Ok and ah sorry, let us say this this is known that well V(0) is 0 and V(x) is positive everywhere else. And ah it is not that ah derivative of x is less than equal to 0 everywhere.

But the derivative is strictly negative. So, if the derivative is strictly negative then we can say that well it is not only stable but it is asymptotically stable. Now, if you remember the difference between stable and asymptotically stable. So, ah with stable we are we will say that well we can define that small region around the origin and we can compute a time. And we can say that well beyond that time, starting from anywhere inside an outer circle, beyond that time, the system is guaranteed to be inside this small inner circle which is definable as a b as a delta using a delta radius like we saw earlier that is your stability. Right. When we say asymptotically stable we said that well it will not only be inside delta. Now, let the system start inside delta and I can guarantee you that with time in the limiting in the limit and the system will approach and reach the origin. Ok So then we will say that x equal to 0 is asymptotically stable.

And the required condition is this that system keeps on dissipating energy. Now, when I say energy, let us understand that this V is what we call from physics is that is an energy function. What V denotes is basically, ah the systems accumulated energy at some point x. And when we take the derivative, it means the system's rate of energy, dissipation. Ok. So, ah for stability, the system will ah try to dissipate it is energy and go to a point in with minimum energy. Right

So, when the derivative is strictly less than 0 it will guide the system to this origin ah where V 0 is defined to be 0. Right. That means the system goes to it is most stable place where it does not it cannot further dissipate an energy. So, in a recap, what the claims are like this, as long as I can define a function V on the state space, which is 0 at the origin, which is positive everywhere else and which is the less than or equal to 0 ah whose derivative, whose time derivative is ah less than or equal to 0 ah over time ah less than or equal to 0 Ah then x equal to then x equal to 0 is stable.

So, let us understand what it means. What is this? It means that let us say I am saying that this is my systems trajectory. And let us say here I am OK. Ah actually we should draw it in a 3D space like this right like the earlier picture. So, let us say it is a 2-dimensional system this is x_1 this x_2 and this V. And right now, I am located, let us say somewhere here at time t_0 and with time ah t_1 ah I should I must be going to a space ah which at which the value of V must be smaller. Right.

Must be smaller or equal but it cannot be larger. So, this cannot happen but this is fine at t_1 and this is at t_0 ok this is the value of V, I am plotting. Ok. So, for that this condition is what we need? That means the system is dissipating energy ah and the system is going to a lower energy state. But if we take this stricter condition and let us say this is my V here. So, wherever I go, I mean as the time progresses, I have a guarantee that V must be going down. Right V must not stall and we must be going down. That means I not only can define that inner ball but I can actually show that well the system will eventually lose all of it is energy and go to this state ah that is V(0) equal to 0 which is it is most I mean, ah state with no energy actually. I mean it cannot dissipate any further. right So, ah that is why this is it is stable point at 0 and the system will be asymptotically stable.

Because it is strictly less than 0 it is always losing the energy, fine. I think we are overshooting time here. So, maybe we will end this lecture and in the next lecture we will start from this point which is the proof of this Lyapunov functions like first theorem that is the direct method. Thank you for your attention.