**Foundations of Cyber Physical Systems**


**Prof. Soumyajit Dey**

**Department of Computer Science and Engineering**

**Indian Institute of Technology, Kharagpur**

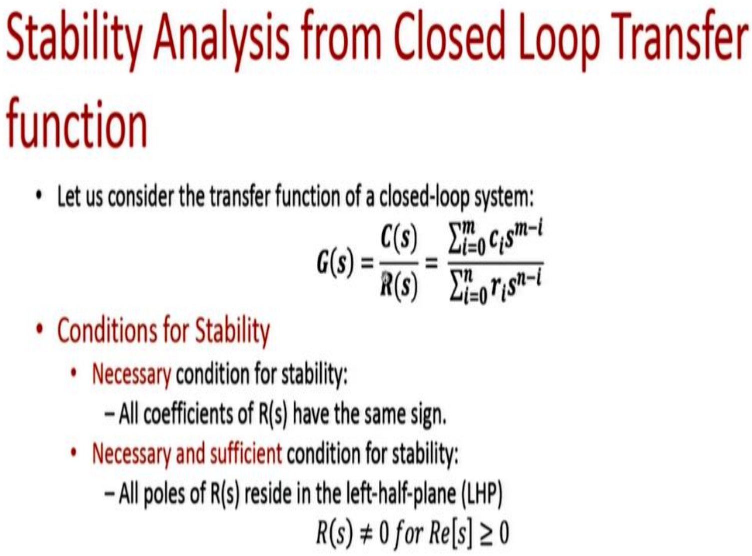

**Module No # 05**

**Lecture No # 23**

**Dynamic System Modeling, Stability, Controller Design (Contd.)**

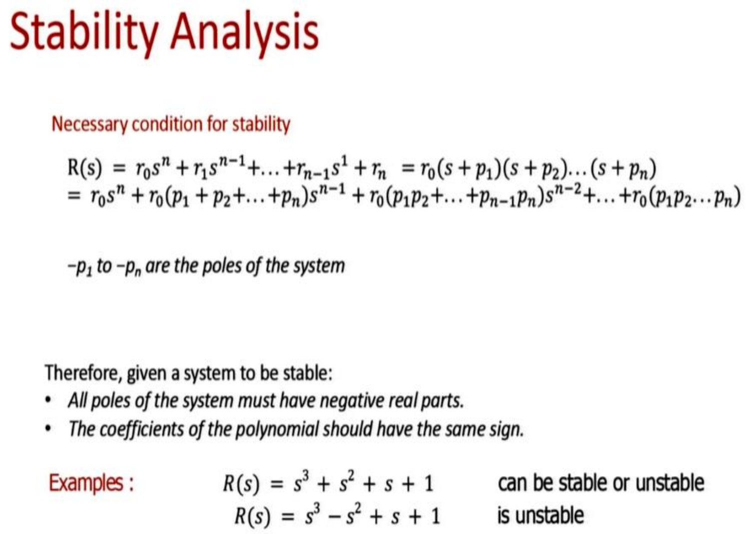
Welcome back to this lecture series on foundation of cyber physical system. So if you remember in the previous lecture we have been talking about how to analyze stability of closed-loop transfer functions. So we will just continue from there.

**(Refer Slide Time: 00:40)**




So let us say you have transfer function of this form a general form G(S) where you have the output by input. This is given by polynomials representing the numerator and the denominator like this. So the necessary condition for stability is that all coefficients of the denominator will have the same sign. But look this is the necessary condition is not the sufficient condition. The sufficient condition is that necessary and sufficient condition is that all poles that we get from this denominator that R(S) they are all located on the left plane that is what we discussed that the poles are located on the left of plane so if you have this satisfied then you have the system as stable and we have also analyzed this based on this few examples of full pole positions and all that.

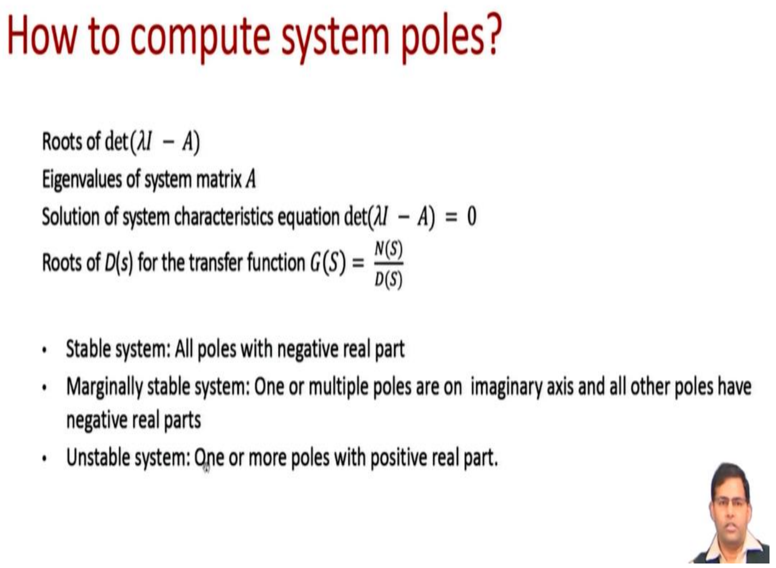
**(Refer Slide Time: 01:37)**




So again repeating the necessary condition would be like this let us say this is your polynomial here right. And from this polynomial if you just create the factor and you find out that well what are the pole positions. These are all poles of the system and for the system to be stable all these poles must have negative real parts and as you can see that for this to happen the necessary condition is that the coefficients of the polynomials this coefficient are 0 to $r_n$ all this. They must all have the same sign so example if you look at this all the coefficient of the same sign. So this you satisfying the necessary condition so I can say that this is stable this can be stable subjected to if we can also prove the sufficient quantity.

So this is because it satisfy the necessary condition all are having the same sign. But then you have to compute the poles and check their positions to be confirmed and because you have to check the sufficient condition. But if you see the next one you say here you do not even have to check the sufficiency condition. That means you do not need to compute the pole position and check whether all of them are on the left top of the s plane. Why? Because it does not even satisfy the necessary condition which is that all the positions are having the same sign because clearly you have some coefficients which are positive and some coefficients like the second one which is negative so this definitely is unstable. So that is why here we are trying to say that why the necessary condition is also helpful to prove that some system is unstable.

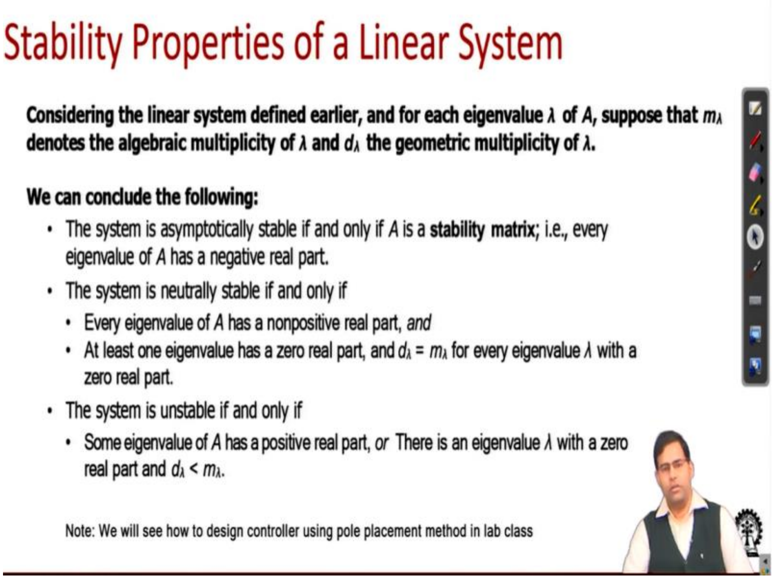
**(Refer Slide Time: 03:16)**




So in general of LTI systems what we can do is suppose you have this representation of the system in this $\dot{X} = x$ form so instead of transfer function if you have this representation but so what you have is you have the system matrix A which is kind of giving you the evolution of the system. Starting from its initial state x is 0. So what you can do is you can take A which kind of represents the closed loop systems dynamics and you compute the Eigen values of the system. The Eigen values of this Matrix is same as the poles of the system and you can just say the whether the system is stable or not based on the nature of the solution without even solving the system model. So again it will just be about the location of the pole.

So the way we compute Eigen value of a matrix is the standard method roots of determinant Lambda I – A where I is the identify matrix. So if you compute the root for this polynomial what you get is the Eigen values of this matrix A. So essentially they are the solutions of this characteristic equation you get the determinant polynomial and you just I mean so you can just set this thing to 0 and for the characteristic equation and then from that you can just compute what are the Eigen values. So the roots that you compute from this characteristic equation they are just the pole positions. So again the condition is same is just that you are starting from A instead of the transfer functions representation here. So all the force must have the negative real part and similarly you have conditions for the unstable and marginally stable systems.

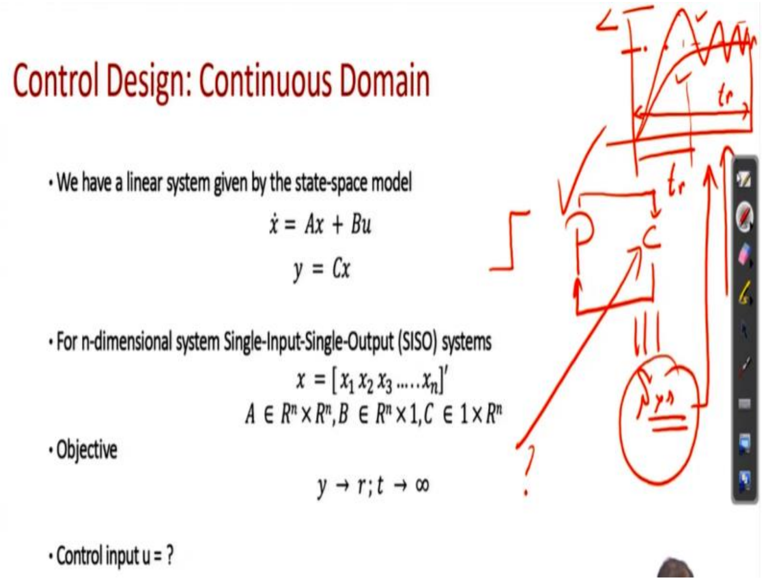
**(Refer Slide Time: 05:06)**




If you are considering the linear system again and you have multiple Eigen values let us say for each Eigen value lambda its algebraic multiplicity is m lambda and the geometric multiplicity is d lambda. Then overall you can conclude this following that the system is asymptotically stable means eventually with t con t approaching infinity the system must settle down towards the reference. Of course for that the necessary and sufficient condition is for A the Eigen values must have the negative real part and the system is neutrally stable if and only for every Eigen value it has a non-positive real part. And at least one Eigen value as a 0 is real part so this is the marginal stability situation and this multiplicity with algebraic and geometric interpretations are same for every Eigen value with has a 0 real part.

So for those Eigen values which have a 0 real part for them these multiplicities must be same and that is about atleast one. So basically it is saying that well at pole is located on the imaginary axis and all the other poles are located on the, atleast one on the imaginary axis and all the others on the left of plane in terms of explained representation. So basically is the same thing but you are interpreting it in in a matrix formulization. And of course the system is unstable if you have some Eigen value that is the positive real part. So that is how you can also can look at it. So that is about how you can create a system model. It can be transfer function based model it can be a state space representation with this matrices and from that you can conclude by looking at this system matrix whether it is stable or unstable.

**(Refer Slide Time: 07:05)**

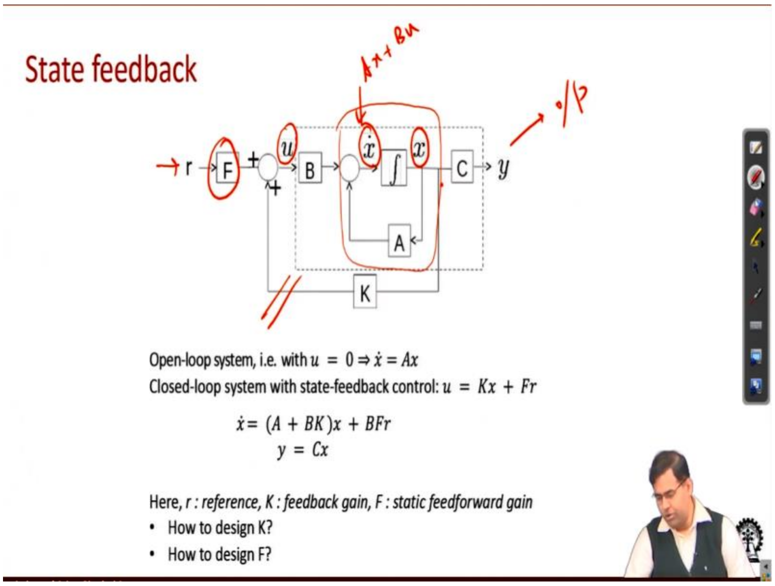



Now suppose we find the system as unstable we want to do the controller design. So the idea is you have this plant and you see that let us say you give an unit step or some response input to this plant and output just wanders around. So overall you have an objective is let us say this is the target reference with an input step you want the plan to settle down here following some trajectory criteria in using and it must meet some settling time pattern. So you want a response like this or maybe a response like this depending on what is your objective you may have a specification that the maximum overshoot should be less than this.

The rise time should be less than some value this or the settling time should be less than some value of this. So this may be your control design criteria and as you can see that just the plan by itself does not meet any of this criteria and the plant by default maybe unstable. So what to do? As we discussed that our key objective in this series of lectures in this week had been that well for we will learn how to model plants mathematically and we will learn some simple techniques of controller design so that when we out put the plant in loop with the controller and derive an equivalent system out of this system is designed to satisfy this requirement. So this is my requirement this is the plant so these are the 2 things given I must design this controller so that the closed loop system satisfies this requirement. There is a problem here we are talking about.

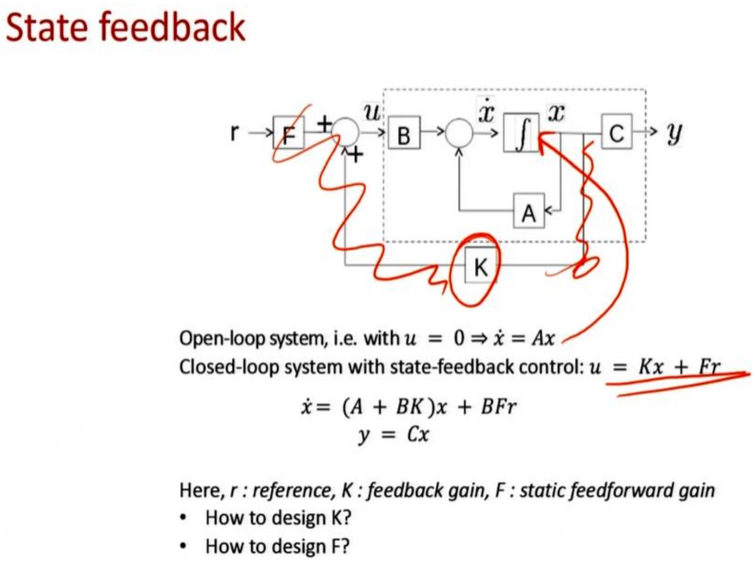
So consider that you have the linear system which is given like this. So this is the state equation and this is the output equation where so you have let us say for n dimensional system you have n dimensional vector of the state and you have the corresponding A B C matrices. And let us say you have an objective that the output must settle down around some reference r with t approaching infinity. The question is how do I design this control input u we are still talking about continuous domain systems. So we will try to design continuous controllers.

**(Refer Slide Time: 09:28)**



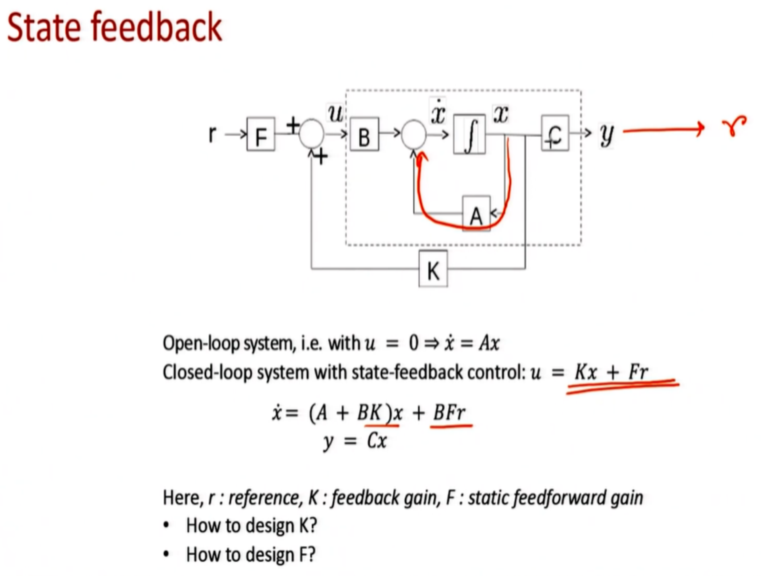

So if you just take a flow diagram based view here so our system is going to look like this. So you have A x + B u which should give you this Ẋ. If you integrate it that's your x if you apply the transformation C that is your output and it is apply this K on x what you get is this value K and then you may have some feed forward gain F here on which the reference is acting. So here you get K x here you get F r and from them you get the overall control signal so that is the thing here. So the idea is when this loop is open you do not have this so all you have is a system Ẋ = A x. So if the loop is open you just have this much happening here, there is no u.

**(Refer Slide Time: 10:49)**



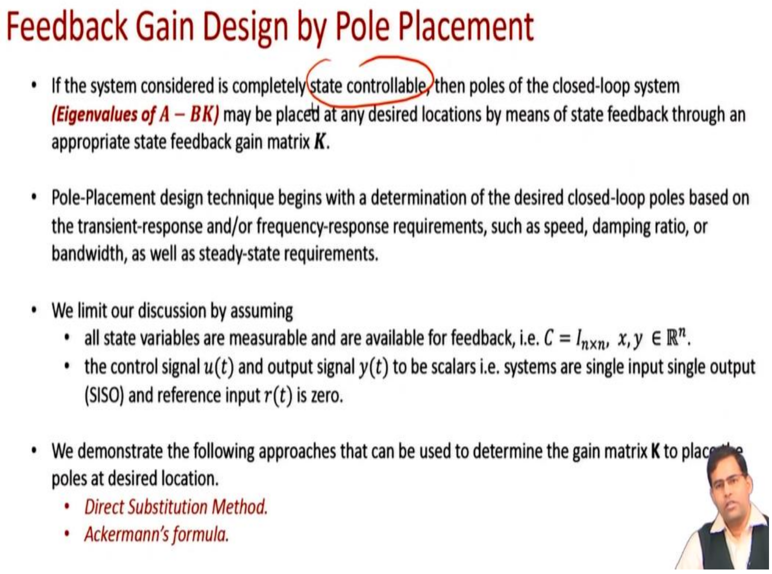
Now let us say we have we want to close this loop and we want to design this controller gain K so that we meet those objectives that we talked about. So you will have this equation. So earlier none of these things were here none of these things were here. So this was pretty much your system that is why Ẋ was just nothing but A x that is was earlier. But now we are bringing in all these additional stuff right here with the gain, feed forward etc.

**(Refer Slide time: 11:24)**



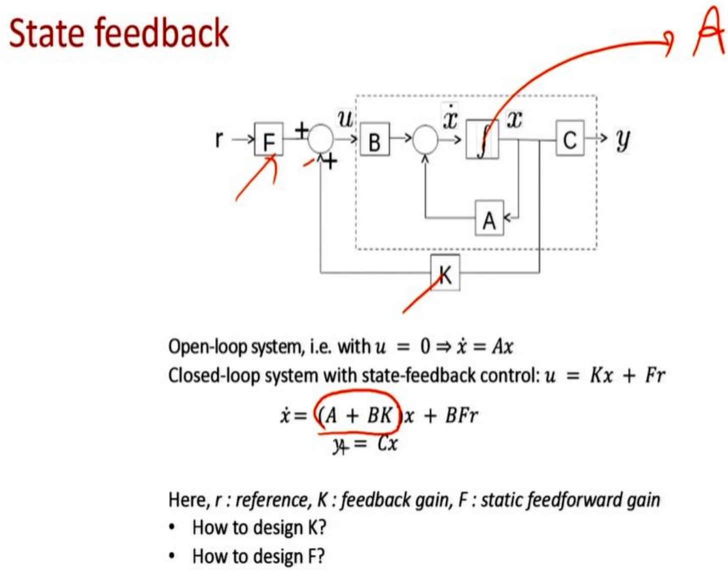

So you want to stabilize this loop so that this response y is approaching the reference r we want this to happen. So if you look at it Ẋ is if you just look at you have equations Ẋ is nothing but A x coming like this and then you have x if you see A x + B u but as we saw that u is nothing but K x + F r. So B u if you see it is B K x + B F r and you have then Ẋ = A + B K multiplied by x + B F r. So that is your dynamical equation and the output y is this C x. So the question is how do I design this K and how do I design this feed forward gain F so that we have this requirement satisfied that eventually the system output must stabilize around r we need to be satisfied.

**(Refer Slide time: 12:30)**



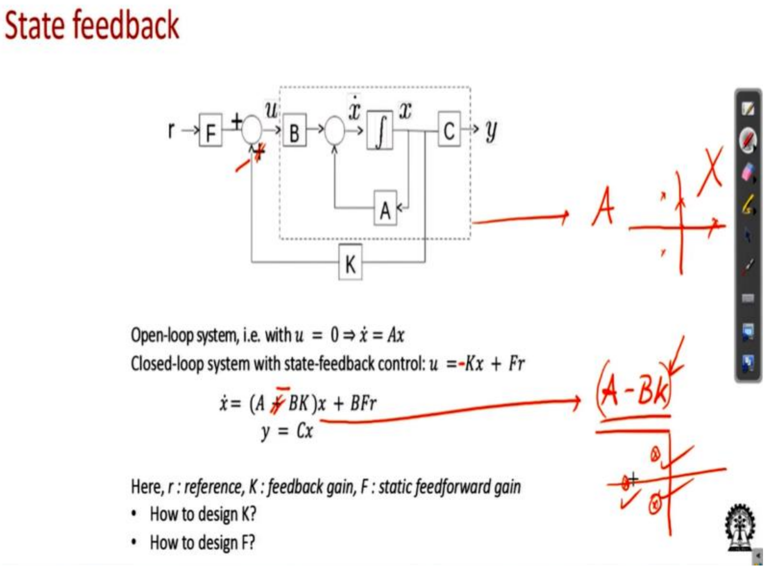

So what we will now study is very well know the most simple way of controller design which is designing this gain K by pole placement technique. So if the system considered is completely state controllable. So we will understand what that means. Then what we need to do is we will design the gain K in such a way that the Eigen values of the closed loop pole must be positioned in a location so that the desired behaviour happens. So that is kind of our target so let us understand what we are really trying to do here.

**(Refer Slide Time: 13:17)**




So you see without this K and f you have this system dynamics defined by A but and that was something I was unhappy with. So I am now bringing in this K and what should be K? And what should be F is in my hand now let us say I have already designed K and F then the resulting dynamics of the system as you can see would be given by this matrix A + B K or I mean depending on how you look at it ideally you can I mean in most books what you will see is this is negative and that is an A – B K.

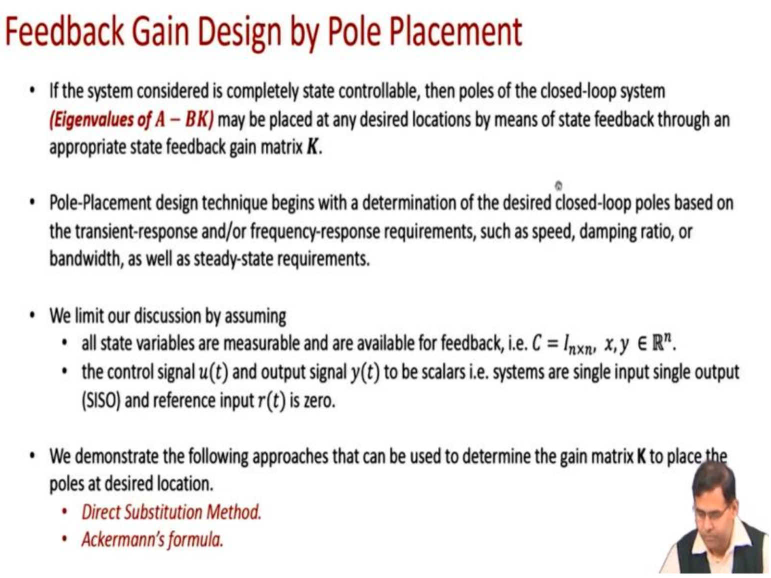
**(Refer Slide Time: 14:11)**




So and actually let us just go by standard convention. So this is –K f + F r this is, so the point I was trying to make is for the open loop the dynamics is dictated by A for the closed loop the dynamics is dictated by A – B K. The trick in this entire situation is you design K in such a way that for the matrix A – B K you should have the poles located at suitable positions so that you get the desired behaviour. So let us say for A which was just given to you which is the unknown plan you may have poles located here let us say here one here etc., and that we giving you some bad behaviour. But now you have an opportunity to modify the closed loop by bringing in K. When you bring in K the closed loop equation is the closed loop matrix is this. This is what is deciding how the system will behave.

So you have an opportunity to design case such that your eventual matrix becomes such that its Eigen values such that the corresponding poles are located at some good positions. Good positions means some positions which satisfies your requirement. So that is the simplest story of pole plus man based control design. So what you do is you decide where you want to place the poles because you know that if the closed loop systems poles are here and then the output behaviour will be nice and that is what I want. So let us figure out what K should be so that for this A – B K the poles are located here and here that all we are doing in pole replacement based controller design nothing else.

**(Refer Slide Time: 16:07)**

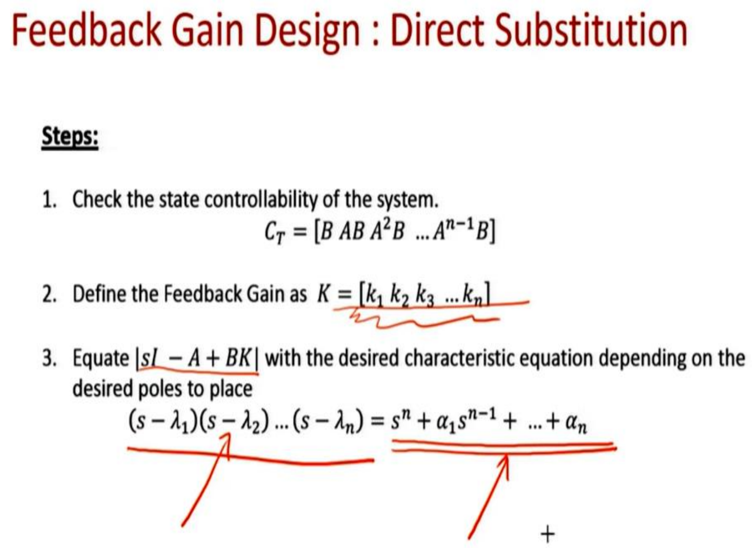



So let us continue with this intuition so the target is to set K so that you get the Eigen values of A – B K located as you want them to be. So this technique begins with the determination of the desired close-loop poles based on the transient response and or the frequency response such as damping ratio, speed etc. So you are given some requirement let us say somebody that you have been told that well you must design the gain K in such a way that the closed loop response should satisfy this rise time this settling time this maximum overshoot etc., So from those things you can have an estimate that well if I place the poles here and here this things I will satisfy then all you do is you bring in the mathematical machinery which will ensure that A – B K Eigen values are located precisely at those positions.

So here in this case we limit our discussion we consider that all states are measureable and they are available for feedback which will just make C as a identity matrix here. So that I mean essentially we are saying that this is the full state feedback system all the x outputs are available for K to operate on. We will soon see in our later lectures that this not what we get in principle and from the output we need to get the estimate of state but we will cover those complex things later on. So for many situations we can also have full state feedback and we just work with it and we can also make one simplistic assumption in certain cases that will the reference input is 0. I mean it may depend or I mean whether I want to design that feed forward gain or if I do not want to design that feed forward gain.

Now we will also see that well how the expression for the feed forward gain can come. So the control signal and the output for the simplicity we consider these to be scalars and the systems are single input single output system and the reference is 0. And we will demonstrate the following approaches for determining this gain matrix K let, one is called the direct substitution method and other is called the Ackerman's formula.

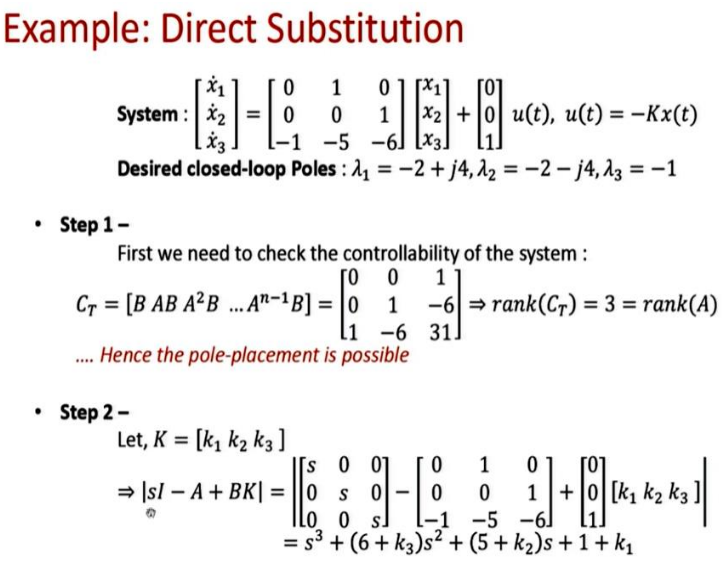
**(Refer Slide Time: 18:24)**




So for direct substitution what you do is you first check this thing what this matrix and look at this state controllability matrix and check its rank. What does this check tell you? It tells that well if this controllability condition is satisfied then I have a guarantee that system is controllable that means it is indeed possible to design a controller which should ensure that in the close-loop the plant and controller combination will be steered eventually we will eventually steered the system in the desired trajectory. And then once this check is done you define the feedback gain using this unknown so your objective is to figure out what are these unknowns and then what you will do is? Well you have already identified what are the desired positions where I am going to put the poles now based on those desired positions you can write this characteristic equation.

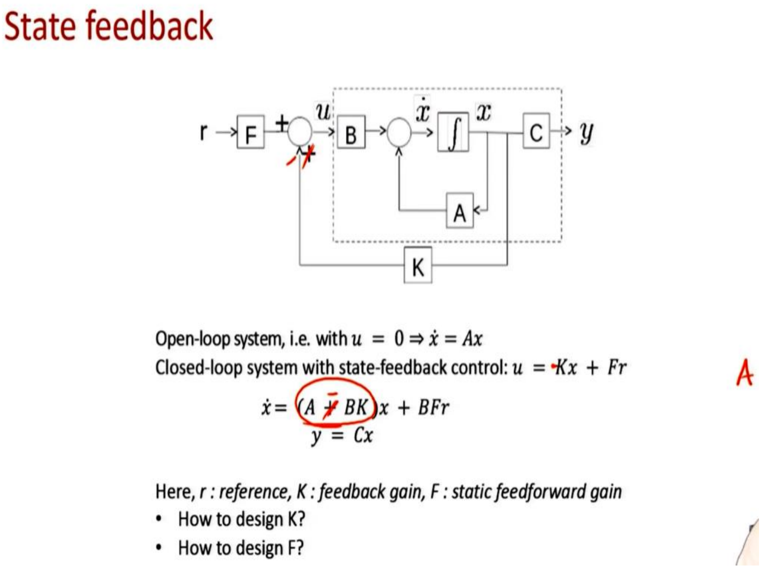
Let us say I have decided that well the poles should be at positions lambda 1, lambda 2, lambda n like that then that means for the target system in the closed loop when I have write the transfer function the denominator of the transfer function basically the desired characteristic equation should be same as this left hand side. And we have this thing we have this unknown gain value now putting those unknown gain values here if we take this determinant we can generate this characteristic equation like this. So this is what I want to happen and this is what I have with sub unknowns, so that is how we will proceed down. So just a bit on controllability we will talk about that first place proceed offer this with an example here.

**(Refer Slide Time: 20:22)**



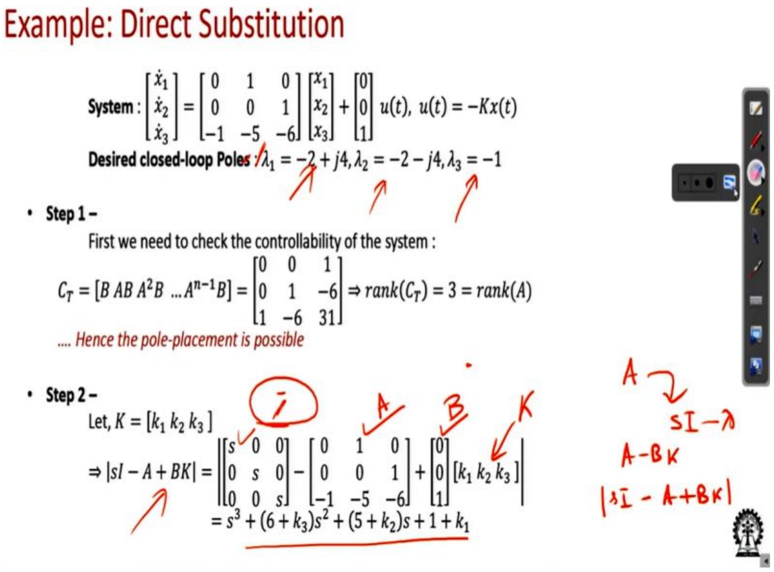

So let us say this is your system. So you have A B given to you and you are generating a single input single output. So u is – K x let us say for this you have the desired pole position at lambda 1, lambda 2 and lambda 3 and the first thing you do is you check the controllability that means you generate this matrix you know what is A and what is B. So from n the dimension of the system is n = 3. So using these values of B A and N you can create this matrix C T and you can check whether the rank of this thing is equal to the dimension of the system. So you can check if the rank of the matrix is equal to 3 which is true then we will say that well the pole placement is indeed possible. Now let us go about doing this things that means equating the characteristic equation with the desired one. So first we have assumed these unknowns K 1, K 2, K 3 to comprise the controller K so with this K which is unknown I can write this determinant right s I – x + B K by the way what is this?

**(Refer Slide Time: 21:33)**



All we are doing is so as we discussed right this is your system and for this system like we said this is the closed loop matrix which is governing the system. So I want to figure out its Eigen values and Eigen values would be nothing but.

**(Refer Slide Time: 21:59)**




So if you remember for A matrix A Eigen values were given by this solutions to the characteristic equation of this thing right. I mean s I – A now in place of that you have A – B K. So for that you have to compute s I – A + B K you want to see well what are the roots of this polynomial. So to compute the roots of this polynomial you have to create the corresponding equation and set it to 0. So that is what you do you take the determinant over this here and see what is the corresponding characteristic polynomial here. So s I becomes this and –A + B and this is your unknown k so this is going to lead you to this kind of a characteristic equation corresponding to this closed loop A – B K matrix this closed matrix. And you want to know what are the roots of this equation so and what you really want is the roots must be this and this is what you want.

**(Refer Slide Time: 23:29)**

## Example: Direct Substitution

$$\Rightarrow s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1 = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)$$
$$= s^3 + 14s^2 + 60s + 200$$
$$\Rightarrow 14 = (6 + k_3) \Rightarrow k_3 = 8,$$
$$60 = 5 + k_2 \Rightarrow k_2 = 55,$$
$$\text{and} \quad 200 = 1 + k_1 \Rightarrow k_1 = 199$$

*.. Hence the gain designed to place the poles of the closed loop systems at the desired location using the Direct Substitution Method is* $\boldsymbol{K} = [199 \;\; 55 \;\; 8]$

So your desire that this characteristic equation have root as follows can be easily achieved. So all you do is well if these are the roots then they together can give me this characteristic equation which is s – lambda 1 s – lambda s – Lambda 3 and this must be equal to the left hand side which is the characteristic equation derived from s I – A + B K. And so they must be equal and if they have to be equal the coefficients of different polynomial terms in S they must be same they must be equated. So Q coefficient must be here like this right. So for example if you see you must have 14 equal to the coefficient of S square because here 14 is the coefficient of S square here. So 6 + k 3 is 14 similarly s coefficient is 60 that is equal to 5 + k 2.So from this you can solve and get k 3 = 8, k 2 = 55 and k 1 equal to something. So essentially what you are getting is you have a system of equations with the number of equations same as the number of variable. And that would give you a unique solution and that would give you these values of k 1 and k 2 and k 3 which gives you essentially the controller.

So the gain does designed to place the poles of the closed loop at the desired location are now computed by equating this characteristic equation of A – B K and its coefficient with the characteristics equation coefficients of the desired closed-loop pole positions and the resulting target denominator of the transfer function. So it is very simple as we can see just to summarize all we did was we are thinking of this target system A – B K right and for this target system we created the symbolic characteristic equation here and for this symbolic characteristic equation we equated that with the form we desire it to be in which was this. And then we just equated coefficients and from these coefficients we calculated what should be the control gain values. So that about it.

**(Refer Slide Time: 26:01)**

## Feedback Gain Design : Ackermann's Formula

- Steps :

  1. Check the controllability and choose the desired closed-loop poles at

  $$[\lambda_1 \, \lambda_2 \, \lambda_3 \, \dots \lambda_n]$$

  2. Using Ackermann's formula we get:

  $$K = -[0\,0\,\dots\,1]\gamma^{-1}H(A)$$
  $$\gamma = [B\;AB\;A^2B\;\dots A^{n-1}B]$$
  $$H(A) = (A-\lambda_1 I)(A-\lambda_2 I)\dots(A-\lambda_n I)$$

  3. Poles of (A+BK) are at $[\lambda_1 \, \lambda_2 \, \lambda_3 \, \dots \lambda_n]$

So that is all about the way in which you do direct substitution and there is also another technique which is basically the same one but it kind of tells you how to do it you using just matrix multiplication standard formula. So the first step is same. So you check the controllability using the similar idea that we give and you choose the desired closed loop poles as these locations like we have discussed here. And then you just apply this Ackerman's formula now what is the formula it says that well K equal to minus of this all zeros then one then you have this gamma inverse times H(A) whereas gamma is nothing but this matrix. Well A B are all given to you and you know the dimension and H(A) is nothing but this matrix. So that is it so essentially this is like automating the previous method. So all you are doing is you are given this lambda right you know A, B and dimension. So you just use this formula and from this formula you will directly get the pole position.

**(Refer Slide Time: 27:09)**

## Example: Ackermann's Formula

**System** : $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -5 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t),\; u(t) = -Kx(t)$

**Desired closed-loop Poles** : $\lambda_1 = -2 + j4, \lambda_2 = -2 - j4, \lambda_3 = -1$

- **Step 1 –**

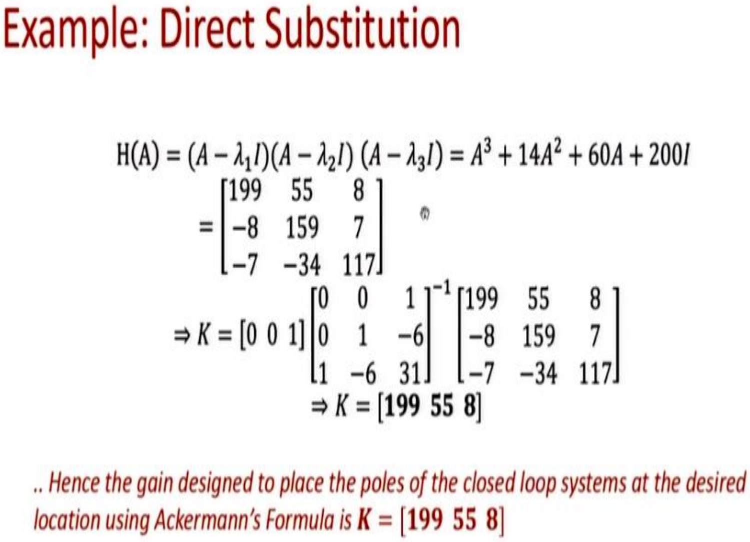
  First we need to check the controllability of the system :

$$\gamma = [B\;AB\;A^2B\;\dots A^{n-1}B] = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -6\\ 1 & -6 & 31\end{bmatrix} \Rightarrow rank(\gamma) = 3 = rank(A)$$

*.... Hence the pole-placement is possible*

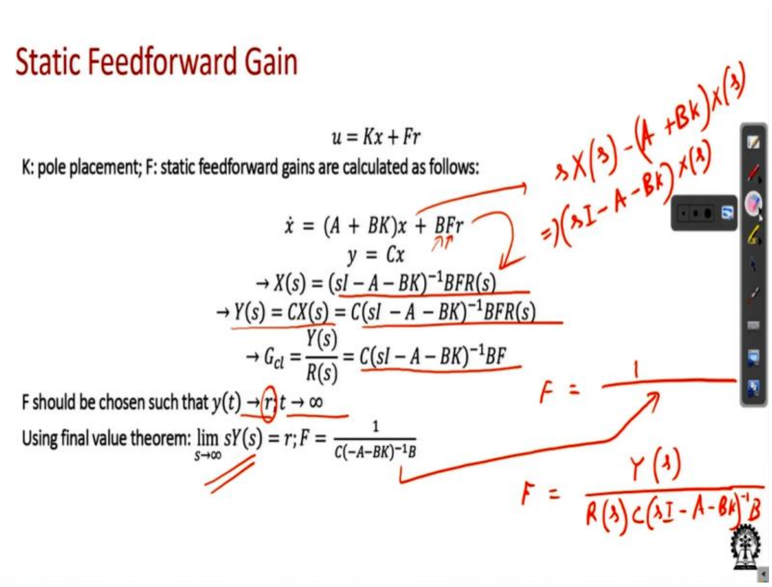
So let us say an example. Suppose this is your system and this is target value of lambda 1, lambda 2 and lambda 3. And well first you do the controllability checks like we discussed and see that the rank is equal to the dimension.

**(Refer Slide Time: 27:27)**




Hence this is indeed possible to design the controller you compute H(A) here and the gamma value is already this one that we have. So H(A) is based on this A and lambda so you get this equation and then K is nothing but this matrix then is followed by gamma inverse and then H(A). So it is essentially automating that entire process of that substitution and you directly get K. So that tells you that well what should be the pole position and you can see that they are matching. So of course some of these examples has been taken from well known lectures and slides available.

**(Refer Slide Time: 28:06)**



So fine and now consider situation where your r is not 0 and you indeed require to design that feed forward gain F. So I mean let us say so this equation is U = K x + F r so your equation $\dot{X}$ is this y equal to this. And you can just solve and figure out what r should be. So let us see what should be if you take Laplace transform here let us see what you get. So you have X s from this it is easy to deduct this things because of course this should give you s X(s) – A – K - B K X(s). If you take this comma s I – A – B this sign really does not matter it depends on how you are modeling the K if you take u = - K x in your initial model just proceed accordingly so do not really worry about this. In our previous example for example in our previous derivation we took that minus because in most books you will see that is the standard that is followed but is just a modeling issue.

So that this is it so once you know that whether the transform is this then then you can just apply s I – A – B k inverse on both sides to get to this line. So once this is the case you know X then you can just see what is  Y. Y is C of X s so that is just this line. So c times s I – A – B k inverse and then you have the B F R(s) because B and s are going to be constant and R is a function of s here. So in general I can say for the closed loop is just the Y by R the output by the reference input. So that function is this. Now the question is how do I choose F you must choose F in such a way that Y which is a function of time it must approach some constant r. The reference with t trending to infinity so you can apply final value theorem here we should tell you that well this limit is nothing but r.

So you already seen this is your expression of Y(s) if you apply this limit it will be and if you equate it to r right. So then eventually what you will get is and r is if you say r as one here so I mean of course this should be equal to r then Y by R from that by using this theorem you can generate F to be equal to this thing here. So you see this form remains that is what you have here and then if you apply the final value theorem you eventually get A f of this form. So in case you really have the reference because if you remember when we did the earlier part so we consider the reference is 0 but of course it is possible that you have a reference signal and in that case you will need this feed forward gain and the way you can design this feed forward gain would be to use this formula like here. So with this we will end the lecture here. Thank you.