**Foundations of Cyber Physical Systems**

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**Module No # 05**

**Lecture No # 22**

**Dynamical System Modeling, Stability, Controller Design**

Hello and welcome back to this lecture series on Foundations of Cyber Physical Systems. So I believe in the last lecture we were talking about the standard responses for second order systems. So just starting from that point. So, what we talked about was that well a typical second order system's transfer function will be like this right.

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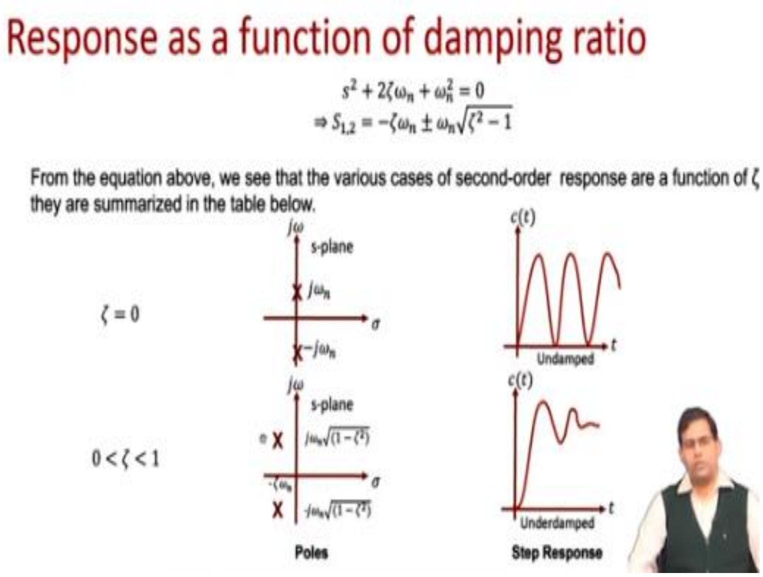


I mean, this is the closed loop transfer function and when you look at the poles of this system, I mean, for the under-damped case, you would have a set of complex conjugate poles right. And for the different cases where the poles will be located otherwise and what will be the different step responses we have already discussed. And so, in general if we just take the responses together and put in a common plot here like this.

So typically, this is how it will be looking like so what we want is that well if you are going to implement a system it should respond fast right. So typically, we will consider this under-damped case. This undamped case is not very useful because well it is an oscillation oscillatory output here.

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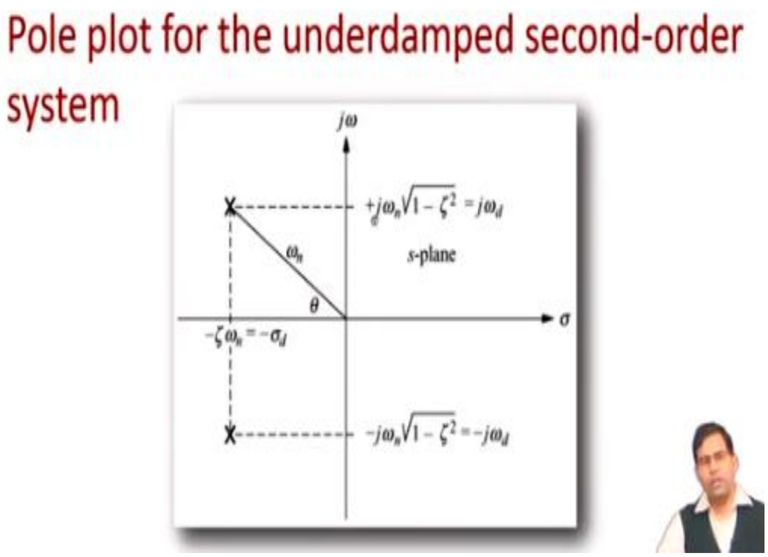


So in general for the under-damped case if we take the polls and for of course for under damped we will have this theta value taken in a way like it is between 0 and 1. So that you have this complex conjugate poles here, so that the imaginary parts magnitudes are same but with opposite signs as you can see and the real part is negative here. So if we just try to correlate like this, here you have these 2 pole positions right with the real part like this minus zeta omega n and the imaginary part given by this values, okay.

And of course, if you just write this equivalent part of omega n times square root of 1 - beta n square if you just write it as omega d, so that would give you the natural frequency of the oscillations.

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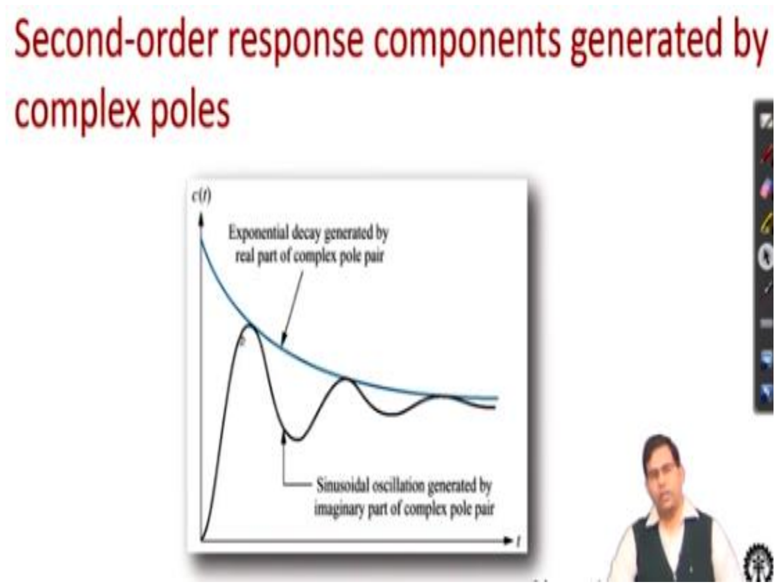
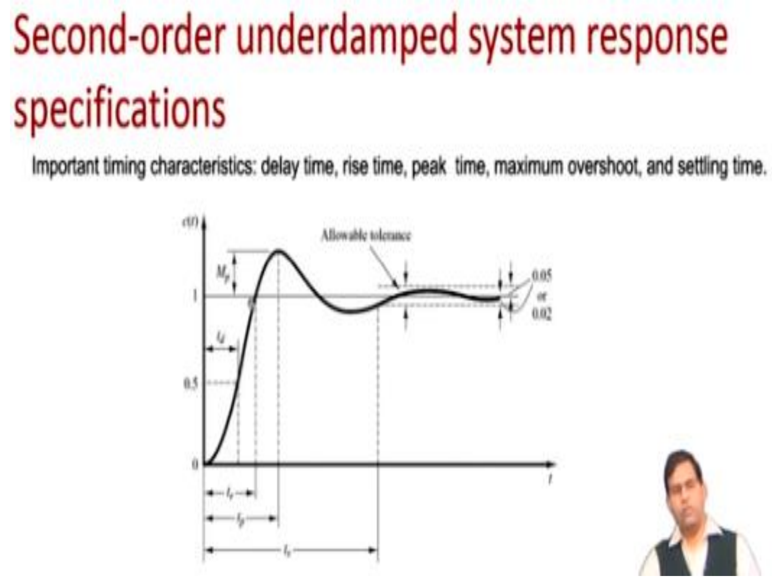



So there are 2 parts here you have this real part which is having this damping ratio here. And you also have this imaginary part. So as we know that the imaginary part is going to be indicating the oscillation right. Because the oscillation frequency will be governed by the imaginary part and the real part will be taking care of, the real part it is having a negative I mean negative damping value. So that is taking care of the fact that well you have an exponential decay on this curve.

So in effect what you have is a sinusoidal oscillation right, which is generated by this imaginary part and if you look at the real part like we have been saying that it is primarily dictating the amount of damping. And you can actually see that if you connect this crest positions of this each of this maximus here if you connect them, you get something like an exponential decay curve and the rate of decay is given is kind of governed by this damping value.

So, if you just like we talked about the response the step response of first order systems, if you just look at the response for this second order system, you can similarly create some specifications like what is the maximum overshoot. Let us say you are targeting this reference 1. And so with respect to 1, the amount of overshoot you have here at the maximum point so that is the maximum overshoot. Let tp be the value denoting the time when this maximum overshoot occurs right. **(Refer Slide Time: 04:21)**




And the rise time is when you are crossing over here the 1 value. And then let us say you have a settling time criteria, which says that well after point I mean by the time when this curve will settle down near 2% or 5% of the target value. So I mean different places we will have different settling time criteria and accordingly you can have this ts design. And also this time by which you are climbing up to this 50% of that target reference, that is like your delay time. So you have delay time, rise time, the peak time, the maximum overshoot that occurs at the peak time,

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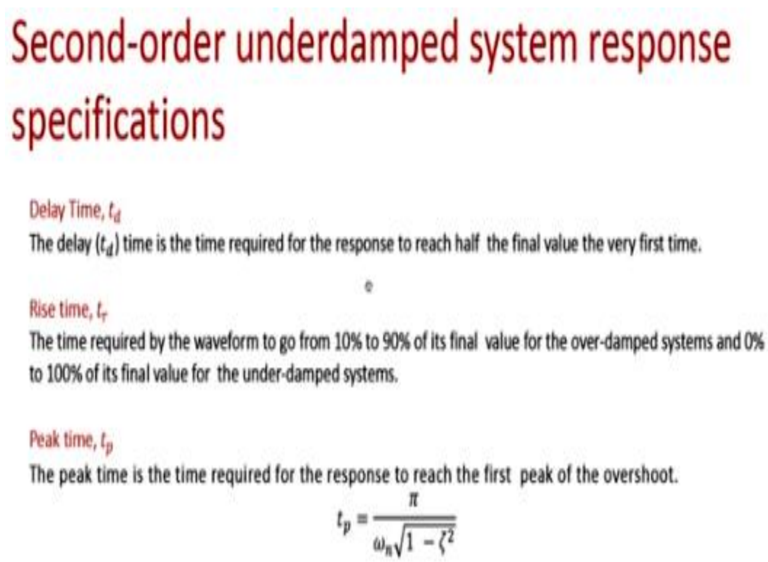


and the settling time specifications and it can be easily derived that well for all this you can actually link up those values with the zeta and omega n. So just to summarize, the delay time is the time required for the response to reach half of the final value the very first time. So as you can see it is this much time the delay here. Then you have the rise time which is the time inside which the value will go from the 10% to 90% of its final value for the over-damped system, and since we are considering under-damped systems here, it is the time by which you are going from 0 to the 100%.

That is the reference value which where you are crossing the reference line up to that much that is the rise time here. So by 100% we mean the final value exactly and for peak time it is the time when you get the first peak of the overshoot and it can be shown for this curve that this occurs exactly at some peak time whose value is related to omega n and zeta by this equation.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

And based on what is your settling time criteria whether it is a 2% settling time, whether you want it to reach and stay, we want the response to reach and stay within 2 percent of the final steady state value or the 5 percent of the final steady state value.

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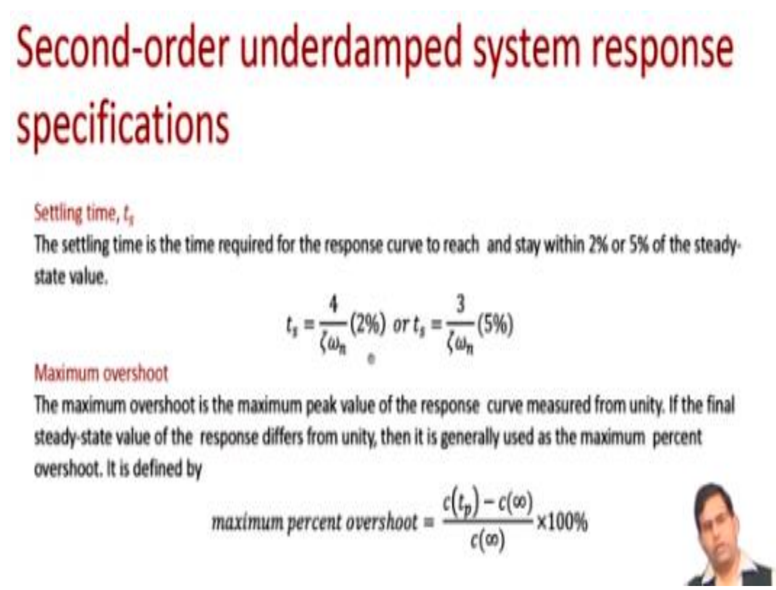


You can actually calculate that that corresponding time and if it is 2 percent then it is 4 by zeta omega n.

$$t_s = \frac{4}{\zeta\omega_n}$$

And if it is 5% it is 3 by zeta omega n.

$$t_s = \frac{3}{\zeta\omega_n}$$

And again and similarly you can also calculate the maximum overshoot which is the maximum peak value of the response. I mean the by the amount by which you cross over from the unit value which is the target reference here. Now if this final reference is going to, if the final steady state value which was unity in this case but in another case if it is differing from unity, then it is generally the maximum percentage overshoot.

I mean if it is unity then you do not have anything here in the denominator but otherwise you can just calculate using this formula.

$$maximum\ percentage\ overshoot = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

That is what is the maximum percentage overshoot.

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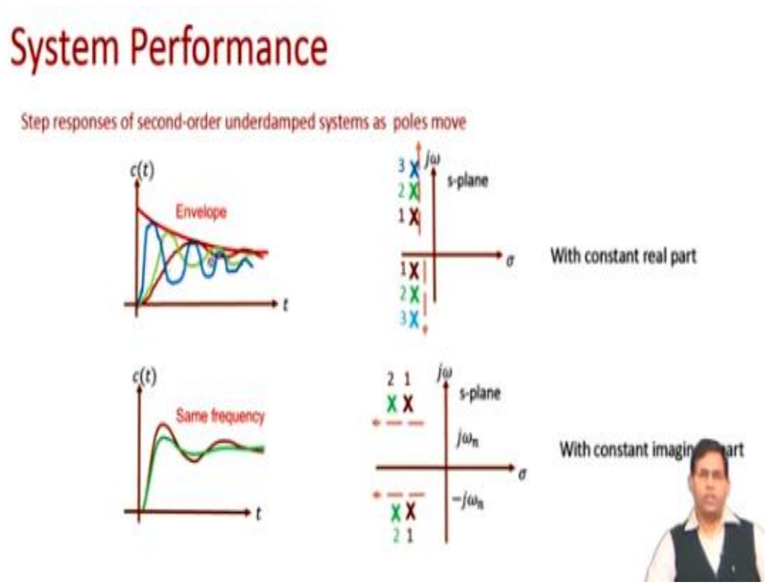


Now let us say you have this curve here of the oscillations, and you are trying to understand that well based on the position of the poles how does the nature of this curves change. So suppose you have the real part which is negative that to be constant and your poles are located at these position 1 or these position 2 or these position 3. That means, the value of the imaginary part let us say it is increasing. So if that increases it would simply mean if you look at the corresponding curve share you are getting the responses with higher amount of oscillatory behavior.

So that means you have you are having more amount of frequent oscillations in the response. But since the real part is same that means the damping envelops, that is exactly same for all these curves. Now let us consider the other thing let us say the imaginary part remains same but let us say the poles are increasing with respect to the, with respect to the real negative real part it. You are going more towards the negative axis. So with 1 you have this situations marked here with x and this is the corresponding curve here right.

And if you look at the other position X for that well you have this this corresponding curve here. Now observe one thing that what we expect that the imaginary parts are same that would really mean that the frequencies remain same that actually is the case. But you want you understand that if the real part is becomes more negative your essentially increasing the damping. So here if you see with green you are going to get a response which is much more damped and that is what is happening right. The green one damps faster, the oscillations are damping faster, although the oscillations are having the same frequency.

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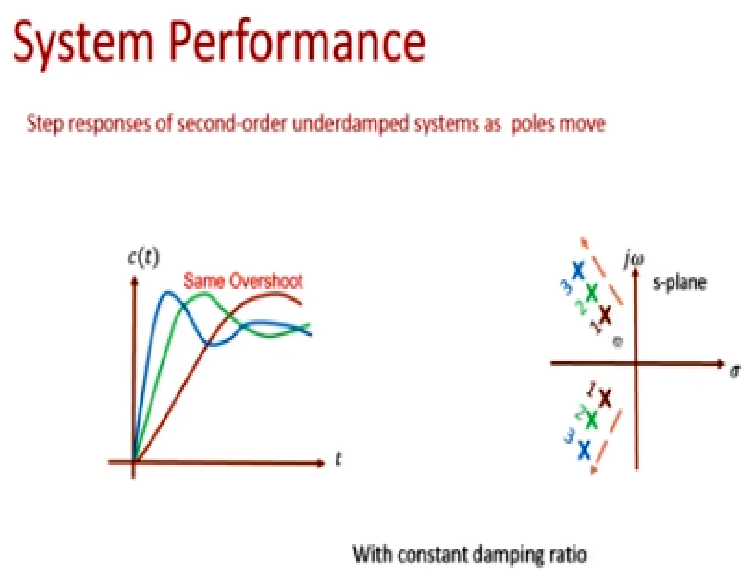


And consider this thing, that let us say you are you are moving along the diagonal. That means well you are changing both of these values together right. The real part and imaginary part both are increasing proportionately right. So in that case your poles are migrating like this. So in this case what you expect is well the damping should increase and the oscillations should also increase right. So if you see here this was the behavior for one with a bit more of damping and the bit more of oscillation you have this.

And with an even more bit of damping and even more bit of oscillation you have this right. So both the damping and the oscillation is increasing as you can see it is going down faster but also the fluctuations are increasing, the amount of fluctuations is increasing. But in all these cases what is happening is the overshoot remains same, because if you try to think that why that should be the case just look at the overshoot condition. So it's like a ratio here right. So that will be one reason for that.

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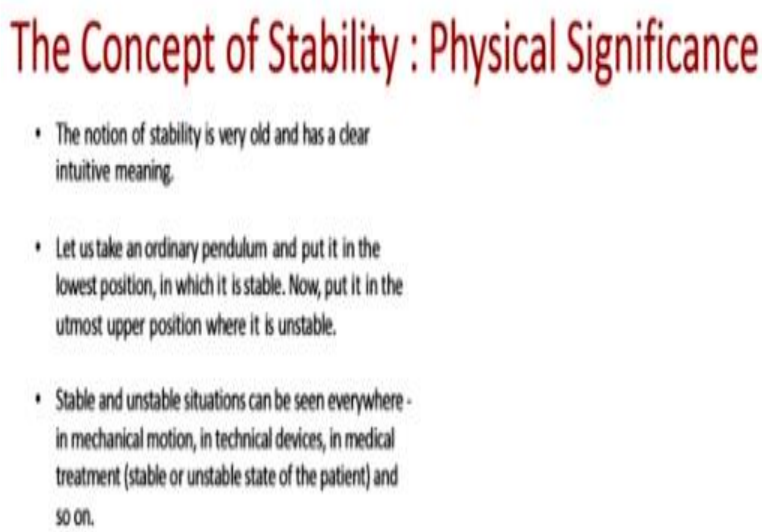



So the next thing we will like to talk about here is well we understand that how first order and second order systems behave. And one good reason why we were talking about first order and second order transfer function suddenly is that it is quite common in case of dynamical systems that a system have complex behavior but you would like to approximate the behavior around that operating point to generate a set of second order, second order or let us say first order approximation of the system.

Now in case you are doing that and you want to analyze from interesting properties of the system this can be one way to go about it. You can check its behavior in terms of, I mean transfer, I mean stability whether the response is oscillating or the response is not oscillating. Whether the response is as you want it to be you can look at those things and decide whether you are happy of the system or you want to modify the system in the in the steps.

Now we saw one thing here which is like there are certain aspects of the system. For example, specifically if you consider these situations. Let us say this situation. So if you have do not have damping here you have an oscillatory behavior right. Or let us say its zeta greater than one its over damp there is no oscillation because there is no imaginary component but it is slow to respond right.

So the point we are trying to make is from these different possible curves of output that we can see, we have an idea that well systems can have oscillatory response system can have a nice damp response like this. But none of these are like mathematical techniques which tell us that well if I am given the state space model of the system or if I am given a transfer functions model of the system, how do I really analyze whether that system is stable or not.

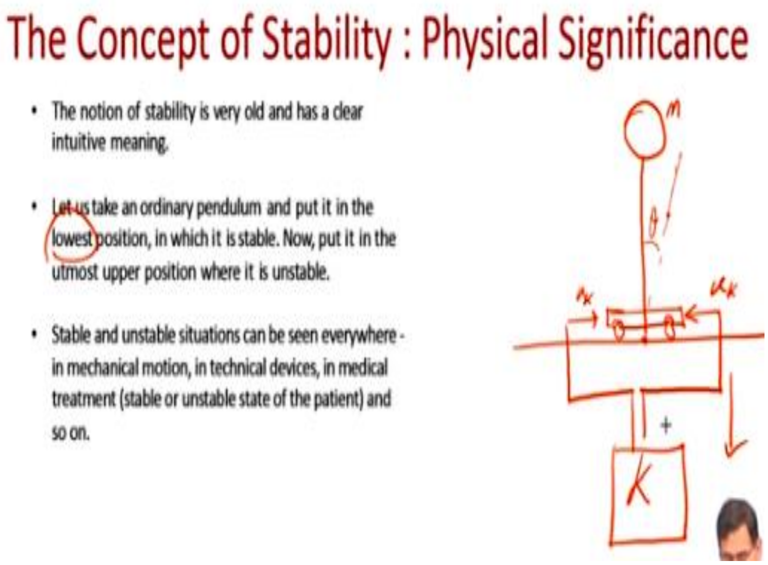
So before getting into this part if you remember, let us go back once. So this was our idea of the state space model and then we said that well what are the advantages of state space models. Just recapitulating a bit you can you can model MIMO systems. You can create time invariant system models, you can model in general linear systems.

You can also have time varying system models using this concept of A, B, C, D matrices to represent the system, where A and B can be a function of time or A, and B, A, B, C, D all of them can be constants. So this is quite a nice and powerful model of representation, a method of system modeling. And then we have by now also talked about the transfer function representation which gives us a Laplace domain representation for LTI systems.

And we have seen that well how this different kind of I mean under certain restrictions like being first order or second order what are the expected response curves that we can get for such systems using standard test signals, like the unit step or some other or the impulse response etc. Now from this, like we have been saying that we understand that based on the pole positions of the system we can we can see that well what can be the responses. Whether there will be oscillations whether it will converge to some point we have seen such intuitive curves.

But like I said I am just repeating we do not have any mathematical criteria yet which tells us that well given the system model, if we can just look at the system order analyze it and we can just directly say that well this will have a stable output or when it will have an oscillatory output or when the output will be not oscillating but rather eventually settling towards some final value. We do not have any such mathematical characterization of such responses.

For this, we bring in this infrastructure of stability analysis. So we will try to define what we mean by stability of such dynamical systems and we will then talk about that well how to analyze given a dynamical model of the system, how to analyze whether the system would be stable or not. So by stable, we mean that the system gives you the system design gives you a guarantee that once the system starts, eventually its response will be steered towards some object, some target position in the state space. **(Refer Slide Time: 15:30)**




So let us understand in a physical manner first. So when we talk about stability the idea is very old. So we can just take up an example of an ordinary pendulum. So let us say we have a free-flowing pendulum here and you just strike it like this with some force and it goes to a position here. So if you see that if you just strike it with your hand like this it will go to a position here and then it eventually would roll back to this position with its top.

Not exactly it will wander about a bit to this side and then again it will come back a bit to this side and then again it will come back but again it will overshoot somewhere here. And here and slowly it will just settle down with the oscillations dying down like this. So this thing entirely is happening due to 2 reasons, you gave a force here u and then the system started moving with that force but its movement got controlled by this gravitational force right.

And due to that it was giving this oscillatory behavior here like that. So when we say by intuition that the system is stable what we mean is it has a default target position. Let us say this angle is theta and the target position is theta = 0 degree, or let us say 0 rad. And unless you are continuously applying some disturbance or external force, this is in stable position it will eventually return back to this position with starting with oscillations and then the oscillations dying down following this kind of an exponential curve here this is the damping right.

And these are the internal oscillations that we just try to create here right. So that is why we can intuitively say that well this is what we can call that it is by default it is a stable system. Now let us just invert this thing. Consider the classical inverted pendulum example. So you have a pendulum like this and let us say it is hinged here. Of course we can understand that this is not a stable position here we have a mass m.

So you have a force acting on it here right, - m g like that. So even if you are able to successfully keep it at this position with minor perturbation, wind or some noise or whatever, it will just get unstable and eventually it will fall to some side. So this by definition would be an unstable system. Now this has got this this is called the classical inverted pendulum model in control theory and there are several variants.

For example you have this cart inverted pendulum. So let us say there are wheels on the cart. This is a mechanical system where it is possible to apply some force. And the control objective here is that you use this force u in a suitable manner. So that you have a control objective that, you will target to make theta equal to 0.

So by default this is an unstable system and that is why you may require designing an external controller K which, at every iteration will inject this control forces to move this curve so that eventually the system stays upright like this. So the point we are trying to make here is this by default is an unstable system.

But if we consider the standard pendulum position with the mass at the lowest level, its the stable system. So we have an idea intuitive idea what we mean by stability the point is for any dynamical system, we can have a control objective which is that it should eventually steer to some set point in the state space. Now the system if it is inherently stable that means, its nature is that in case there are perturbations, the system will steer around a bit around that set point, but eventually by its own it will go to the set point.

But if it is the inherently unstable system the idea would be that if it is being perturbed slightly it will slowly start building oscillations and it will go far away from that set point, say that set point there right. So that would be an inherently unstable system like this and for the such systems we will learn to build controller so that then with the plant and the controller together your objective would be to make the system stable. So this still is not a very formal and mathematical representation of what indeed stability is.

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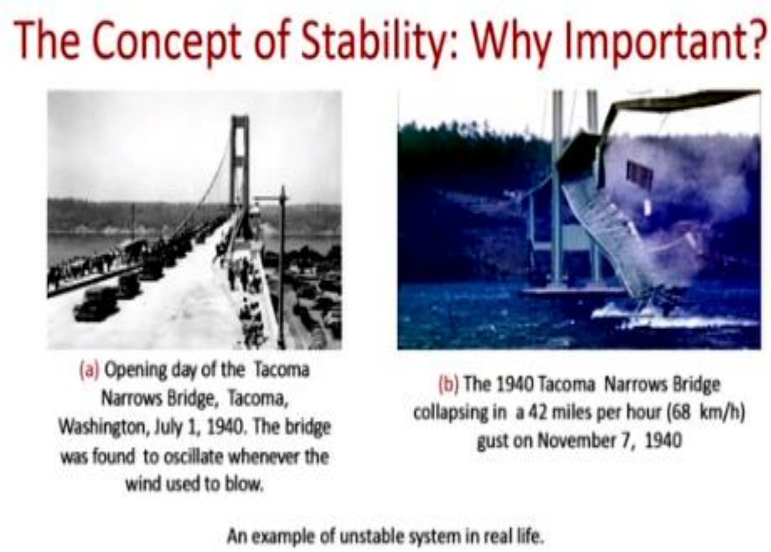

So let us continue and see. For example here you have some illustrative examples taken from some well-known control theoretic literature. So we are showing that well there was a bridge built in Washington and it was found to oscillate whenever the whenever wind used to blow and eventually that I mean the bridge collapsed, because the design of the system was not inherently stable. The designers failed to make the system stable with respect to perturbations from wind noise and other weather conditions.

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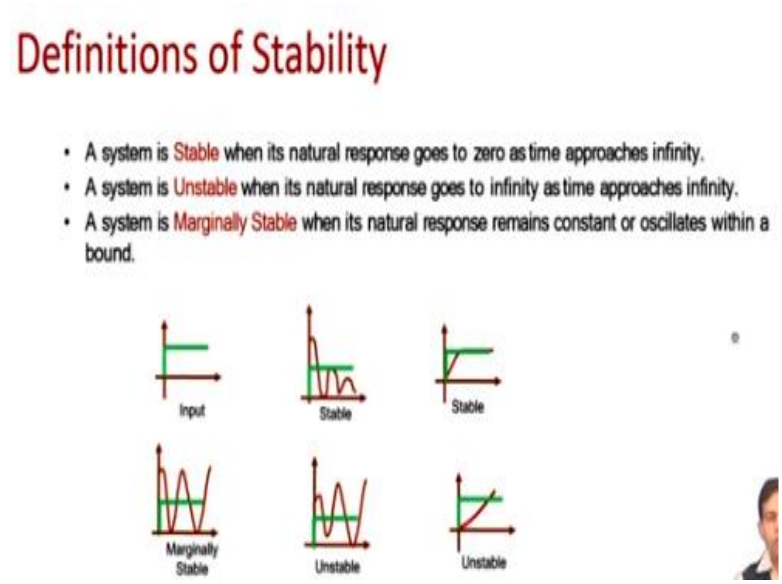


So based on this we can have this kind of a pictorial description. We can say that well a system is stable when its natural response goes to 0 as the time approaches infinity. So if you remember from our previous discussions we said that the system's time response will have 2 components. There is the natural response and there is also the steady state thing right. So when this natural response, the initial oscillations do not build up and it goes dies down and only the steady state behavior remains.

We would like to say that the system is stable. But if it is such that the natural response saturates towards infinity, I mean, it is never trying to come back to the steady state to some set point, we like to say that this is the unstable system. And we will say that it is a marginally stable system, when its natural response is constant that means well it is never settling down around the reference, but it is neither is it moving towards infinity as with time increasing, but rather it has a bounded oscillation.

So suppose you have given a unit step and the system if you have got this so here we are plotting only the natural response part. And we are saying that the natural response is dying down. So that is a stable behavior. Even this is a stable behavior because this is under-damped, I mean over-damped case. Eventually you are rising up and you are able to follow the trajectory. That is also stable but if you have something like this that where the natural response is keeping on increasing is unstable.

And this is marginally stable because this curved trajectory tells you that well you are never going to settle down it will just continue like this, but that oscillation is always bounded so it's marginally stable. And even here you can have the output going beyond the reference and slowly with time this difference is increasing even without oscillation. So that is also an unstable case. So that is how pictorially we can kind of sum this thing up, that what really do we mean by stability of a system.

So if we now try to formalize a bit, we can say that well a system is stable in terms of bounded inputs bounded outputs or BIBO stable. If for bounded input disturbance that we provide to the system the oscillation that we observe in the output is also bounded. So that would be bounded input bounded output stable system. Absolute stability means this eventually this behavior that means eventually I mean the output response will be bounded and output response will be damped and eventually the output will just settle around whatever is the target position.

And then we also have this concept of relative stability that the degree of stability. So that would be measured by observing the relative real part of each root. So for example, suppose we have 2 systems and you I mean you see both of the systems will we for let us say for 2 systems in this diagram we are claiming that r2 is relatively more stable than the pair of roots which are labelled as r1.

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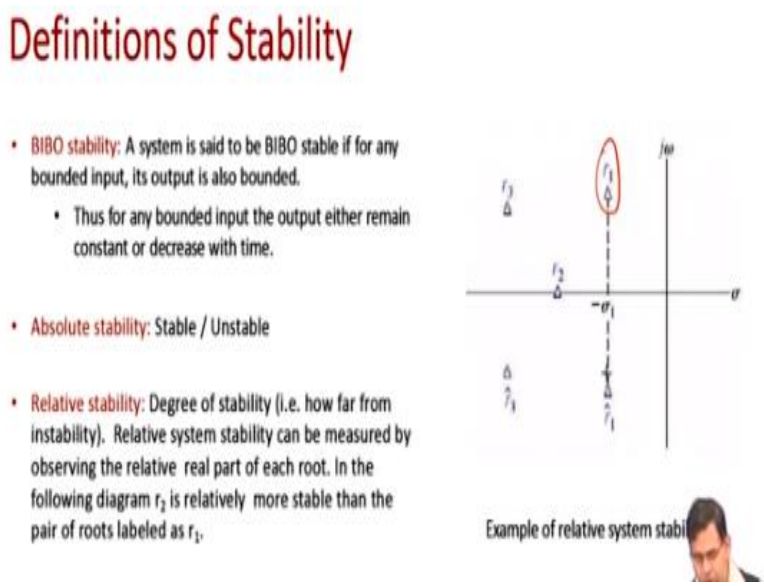


So suppose you have a system which has got for the roots of the denominator polynomial at these positions. And let us say you have another system where you have the pole located here. So the idea here is that where this is more into the left half plane, the negative real part is more. So the system would have more damping and the system should settle down faster towards its eventual reference value, and it is kind of more stable because it may be having lesser oscillations and the damping is larger here.

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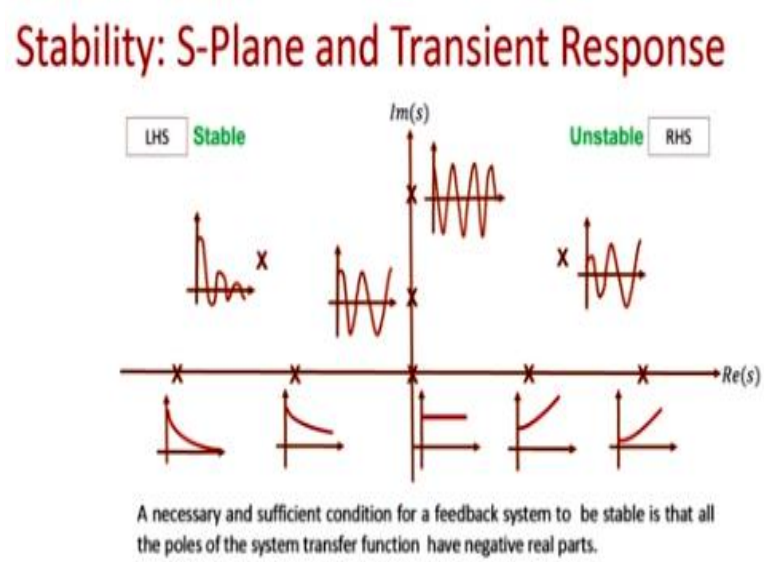


So the way we characterize stability is something like this, that well, suppose you are given this transfer function and you have figured out what are the position of the poles. If you see that all the poles have the real parts to be negative, we will say that the system is stable. And if you have some real parts of poles to be positive, we will say that the system is unstable. And this can really be observed here from our previous analysis and examples also, because whenever you are having this all these poles with negative real parts, that means, you are having a damping on the output so that even with oscillations or without oscillations based on the position of the other poles or the same pole with respect to the imaginary part, you may or may not have oscillations but the damping is ensuring that eventually you are moving towards the reference. But in case you are having some pole with a positive real part, that would means you are having poles in this side of this s plane and that would really mean that well you do not have damping and your outputs are will always be building up and the deviation from the reference point is going on to increase like this.

So there is no chance that your system and in the steady state will settle down towards some reference. It may happen with also without oscillations like here or it may happen with oscillations like here depending on whether the pole has an imaginary part or not. But as long as the pole has a positive real part you are not having damping So this is the situation is unstable. If you have poles with no real part but only the imaginary part here, then you will have this kind of a marginally stable scenario like this because you are going to have oscillations and due to 0 damping the oscillations are never going to die down.

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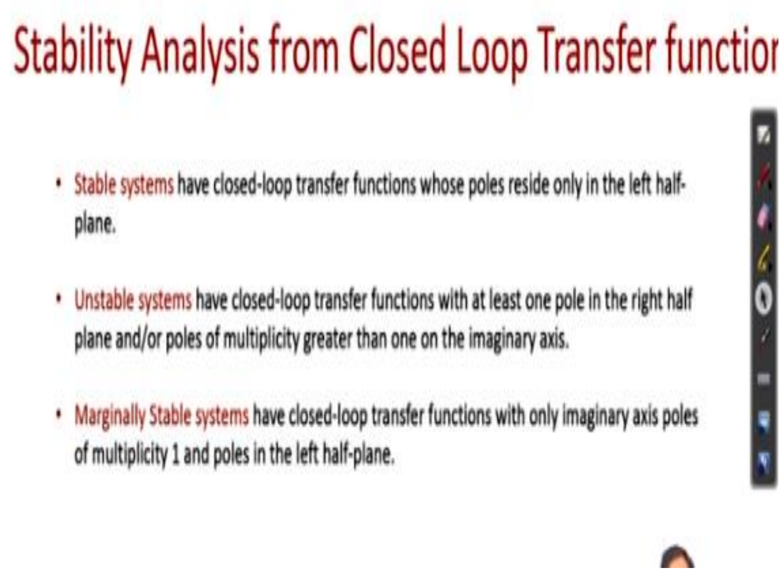


So we can say in a nutshell that stable systems have closed loop transfer functions whose poles reside only in the left half plane. Unstable systems will have closed loop transfer functions, where at least one pole will be in the right half plane or the or there may be more poles, that means, at the same position. So poles of multiplicity greater than one are on the imaginary axis. So either you have at least one pole in the right half plane and or or, you have poles of multiplicity greater than one which are on the imaginary axis.

And you have marginally stable systems, if with only imaginary axis poles of multiplicity one, and you have other poles on the left half plane. So you have marginal stability if the transfer function has got the poles in the left half plane, and other poles which are only having I mean their values which are which where the real part is 0 and they are lying exactly on the imaginary axis. So that's how we can classify stable, unstable and marginally stable systems and the next thing we will talk about is stability of closed loop transfer functions.

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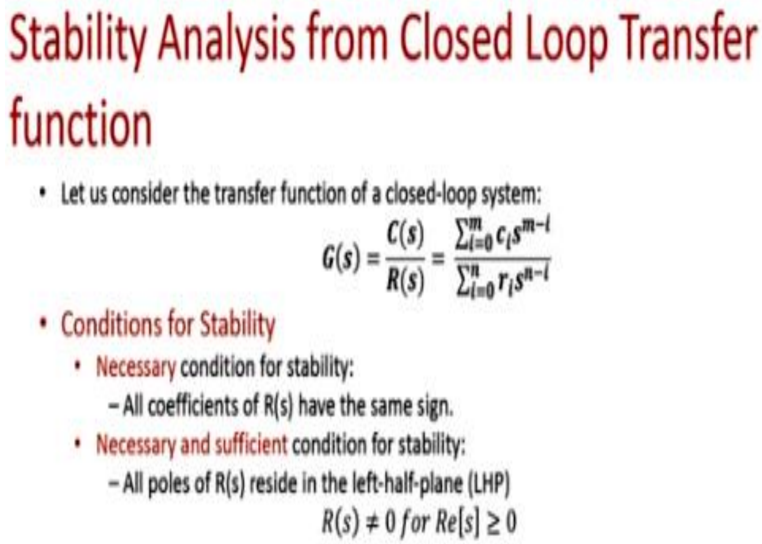


So we will end this lecture right here and continue in the next class from this point. Thank you.