Foundation of Cyber Physical Systems

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Module No # 05

Lecture No # 21

Dynamical System Modeling, Stability, Controller Design (Contd.)

Hello and welcome back to this lecture series on Foundations of Cyber Physical Systems.

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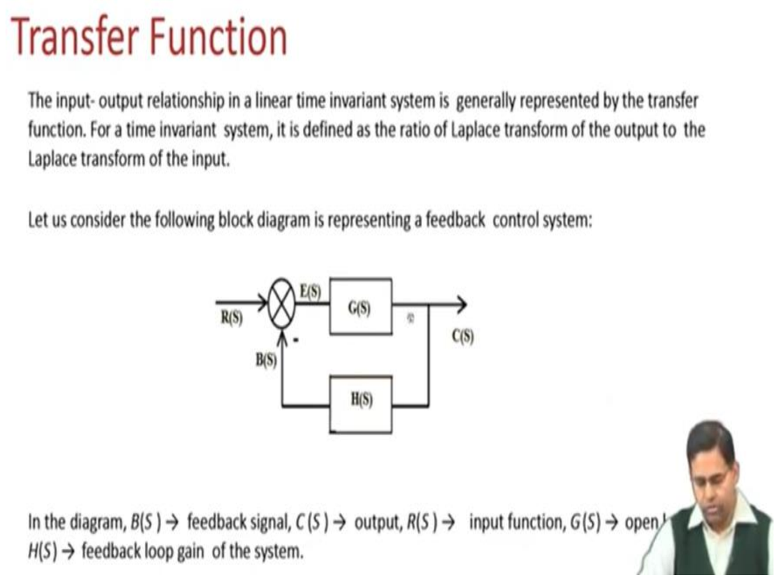

So if you remember in the last lecture we have been talking about this transfer function view of the system. So let us say if you see this picture, you have B(S) which is the feedback signal C(S). C(S) is the Laplace transformed output signal. R again is the Laplace transformed reference input function and E(S) is of course this R(S) – B(S) here, and G is the open loop gain and this is the open loop gain.

Assuming there is no controller and there is an output and then assuming there is no control and there is no feedback and with feedback here from the output measurements, we have the feedback loop gain or the actual controller here. So as we can study this diagram and we can quickly write control equation, the equations which represent the interrelation between them I think that is quite easy to figure out and you will find it in all standard control theory books here.

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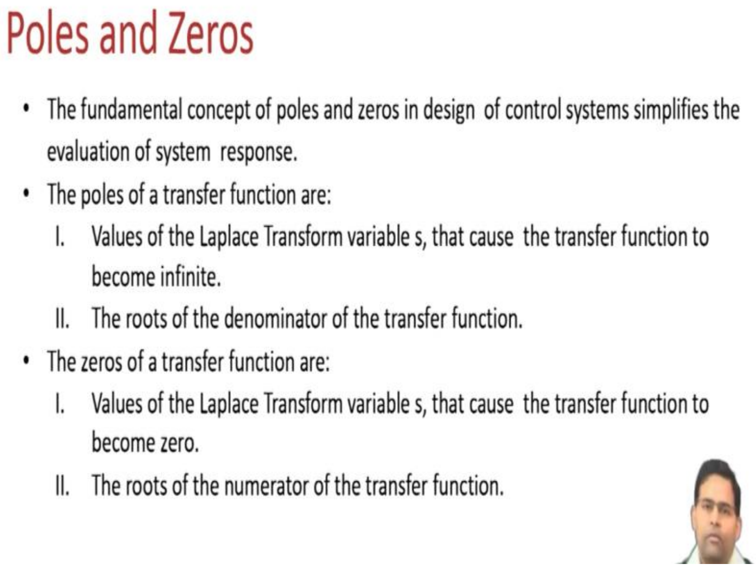

Now coming to a way of studying such transfer functions. So this leads us to these concepts of poles and zeros which are very fundamental in the design of control systems and figuring out how the system response is typically going to look like. So what really are these. So the poles of a transfer function are the values of the Laplace transform variable that cause the transfer function to become infinite and basically if you have a transfer function with a numerator and the denominator.

So the denominator considering that as a polynomial over the Laplace variable S, the roots of the denominator are the pole. And similarly, if the roots of the numerator are the zeros, because if you set those as the values of S, the functions will the transfer function will evaluate to 0.

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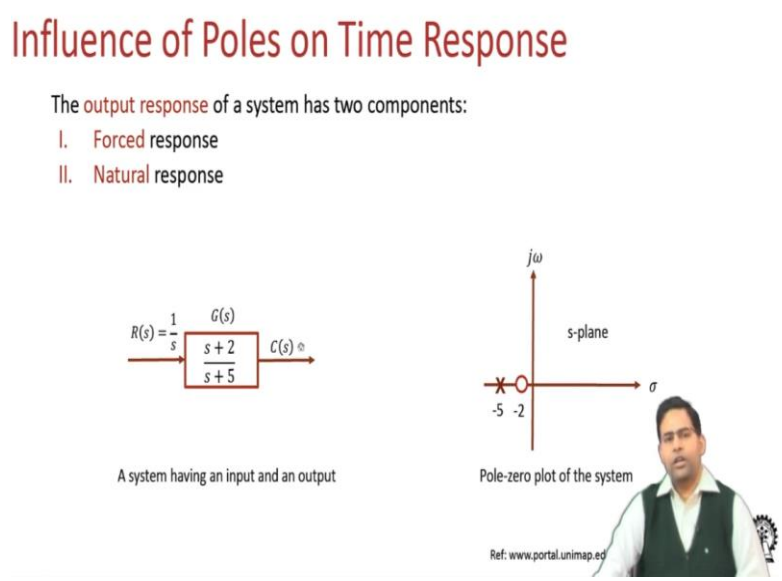

So for example in this one if let us consider a transfer function s +2 by s +5 so you have a pole at s = -5. And you have a zero at S equal to, so this is your location of pole. They are conventionally marked with a cross and the zero is marked with a circle. So s is equal to -2 you have a zero. So this is your s-plane representation. Now we will try to link up this idea of transfer function based system representation and how based on the standard test, this standard test signals, these systems create a time response and how those time responses depend on the location of the poles and zeros. We are going to look at this relationship right here. So like we said that the output response has this component the forced and the natural response here.

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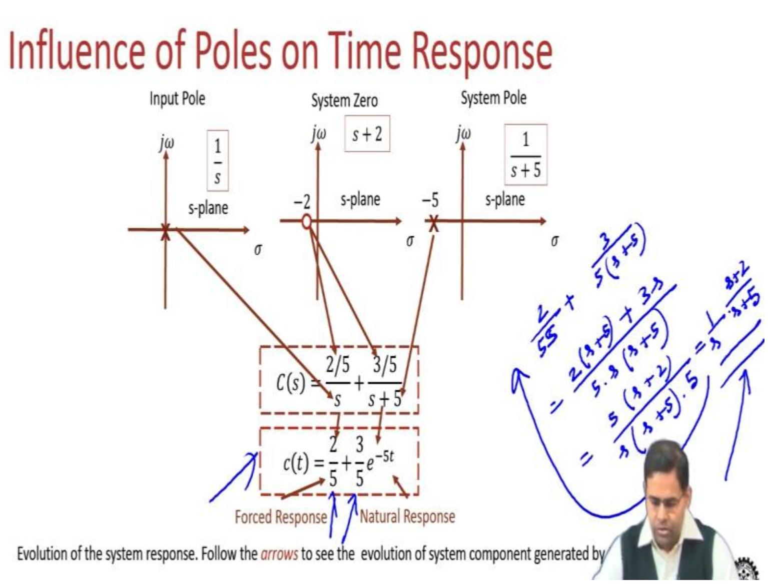

So let us look at the response that is expected here so you have this transfer function and you have this as you can see it is a unit impulse input.

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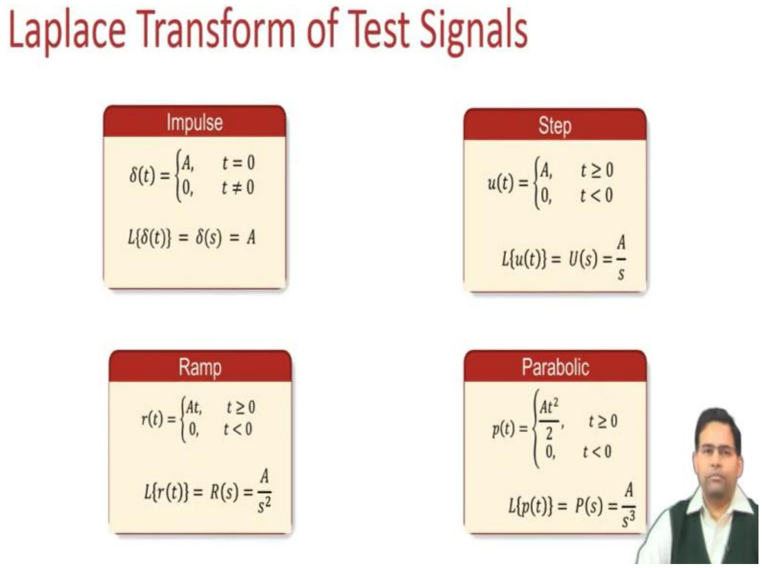

If you want to verify here you have so sorry you have a unit step input 1 by s. So let us see what we expect here. So if you do this thing that is. So this is what you have the, convolution in the time domain will become just a multiplication in the Laplacian domain here. So you should be able to write this in this form. Because s times 5 and here, so if you see, you are looking at this as your system and if you try to see what is going to happen.

So this is your input which is the unit impulse. So that is having a new input pole here and this is where you have the input system zero and this is where you have the system pole. So in this slide, if you see, we are considering what that there is this unit impulse which is being given as an input to this transfer function. So effectively what you have is this in the Laplace domain, the unit impulses getting multiplied by the transfer function.

Because in the Laplace domain your convolution in the time domain effectively must be a Laplace function in a multiplication of this Laplace in the time domain all multiplication in the Laplace domain here. So if we just transform it in this way, so this is what we can write. We are trying to create a separation between the 2 responses that are there so that, with that objective if you just write it like this.

So you have this you introduce a 5 here and then if you see you can just separate it out like this. You have 5 s +10 here and you can write that as well 2 s +5 +3 s. So you have effectively 5 s +10. And then what you will get is this independent form, which is 2 fifth 2 by 5 by S and the other is this 3 by 5 by s +5. Now question is why we are doing this, where we are writing trying to write this multiplication in this specifics form so that now if you take the Laplace inverse and come back to the time domain, you are able to separate it out you are able to separate out the response with the time variant and time invariant parts. You get a 2 by 5 and you get a 3 by 5 times e to the power -5 t so this is your transient response, which will die down with time and this is the part which will remain. And so that would be your steady state response here.

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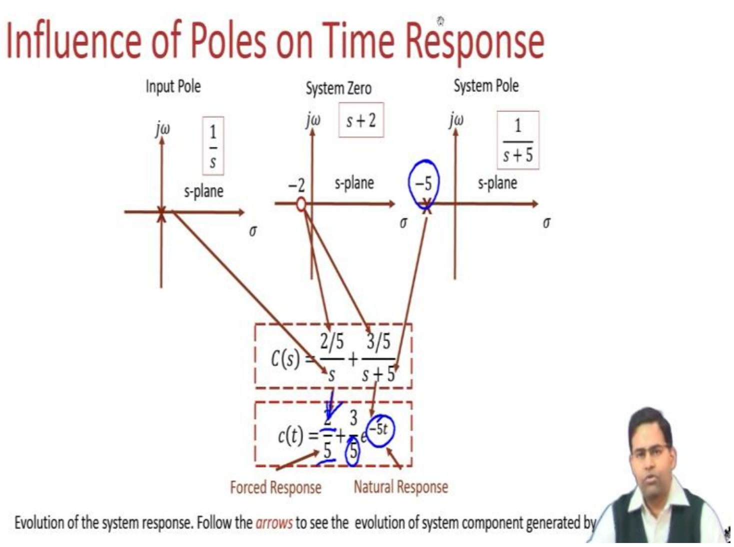

So based on this derivation we can just say that well we started with the system as a simple transfer function we gave it a unit impulse, unit step input. And based on that if I just do the multiplication in the Laplace domain; and do these basic transformations and then come back to the time domain. We get a steady state input and a transient input here like that. And why we are doing this is if you observe these values of this transient and steady state response, you can see that they are kind of being dictated by this location of the poles and zeros.

Because you see this location of the pole and the location of the zero is kind of dictating the value of the steady state and the transit response here. So if we just use a further magnification here, look at the location of the pole. You can say that this is also determining how fast this transient response is going to die down. And how much damping should be there in the response. And it is also determining that I mean both of them together are determining that well what should be the value of the steady state response. So I am trying to just impress upon you that how the location of the pole and 0 are going to give you an idea that what should be your response for the transfer function.

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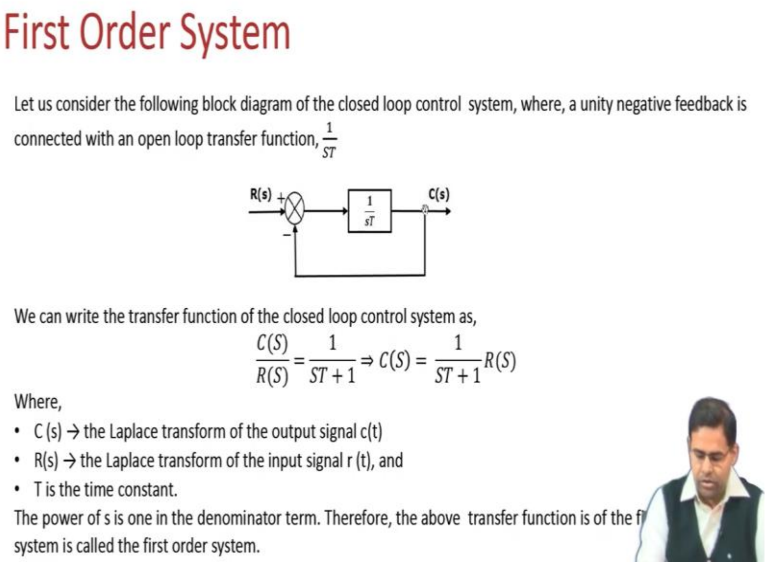

Now with these basic ideas let us start analyzing transfer functions from starting from the simplest cases. Let us start with the first order transfer function. So we consider this first order transfer function 1 by ST with a unit negative feedback. So if you try to deduce the closed loop transfer, so, it would be something like this so C(S) by R(S) is this well, how this is coming if you just want to figure it out it is quite simple actually. So here you C(S). So the way it typically comes is exactly the way you can actually write the open loop, the closed loop transfer function, or the closed loop gain for this given the open loop gain G(S), if you just apply that same method here.

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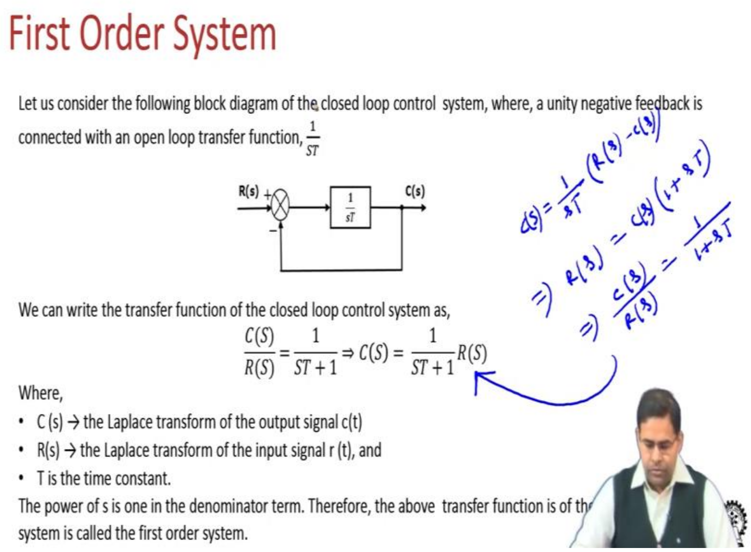

So by doing such standard manipulations, you can actually figure out what is the closed loop transfer function here which should give you this.

$$\frac{C(s)}{R(s)} = \frac{1}{ST+1}$$

This is the first order system as you can see, because you have only, I mean you are not having a second differentiation of x here. A second or second its only the first differential. And that is why you only get the term ST when you do the Laplace transform here. So that is the point the power of s is 1 and so this is the first order system.

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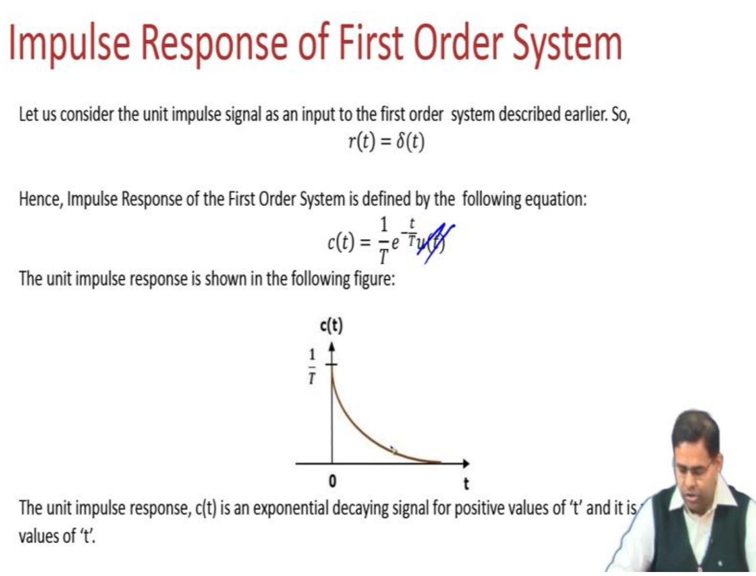

Let us now try to see what would be it is time response so if you just take R(S) to be let us say this unit impulse function delta t. So then R(S) is then r(t) is their unit impulse then R(S) is of course 1 and if I just evaluate c(t) in the time domain from 1 by ST +1 you take a inverse Laplace all you get is this.

$$c(t) = \frac{1}{T}e^{-\frac{t}{T}}$$

So 1 by T, e to the power minus t by T. So that is what you have. And well this u(t) should not be here. So this is this is your response you can see that it is like a capacitor discharge curve is starting from the amplitude 1 by T it is exponential is decaying. So that is about how you get a first order response and that means in this case your transient response decrease to 0 and the steady state response is 0 anyway.

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Step Response of First Order System

Let us consider the unit step signal as an input to the same first order system. So,

$$r(t) = u(t)$$

Hence, unit step response of the First Order System is defined by the following equation:

$$c(t) = (1 - e^{-\frac{t}{T}})u(t)$$

The transient term in the unit step response is –

$$c_{ts}(t) = -e^{-\frac{t}{T}}u(t)$$

The steady state term in the unit step response is –

$$c_{ss}(t) = u(t)$$

The value of the unit step response, c(t) is zero when t = 0 and (-)ve. It is gradually increasing from zero value and finally reaches to one in steady state. Therefore, the steady state value depends on the magnitude of the input.

If you do a first order response, so and then what will happen is, well you will have you will have this R(S) as 1 by S. So effectively you will have a first order system which would be given by this function. So this will evaluate to 1 here, the unit time, and then the transient response is given by this and the steady state response is given by this. So this is 1 of course because it is a unit function here. So what will happen is eventually the value will be 0 at t = 0, that at t = 0 the value will be 0. And with time when the transient will die out the value will be 1. So you are gradually increasing from the 0 value to this final value of 1 in the steady state. And the time required for that increase would be will be determined this value of t here.

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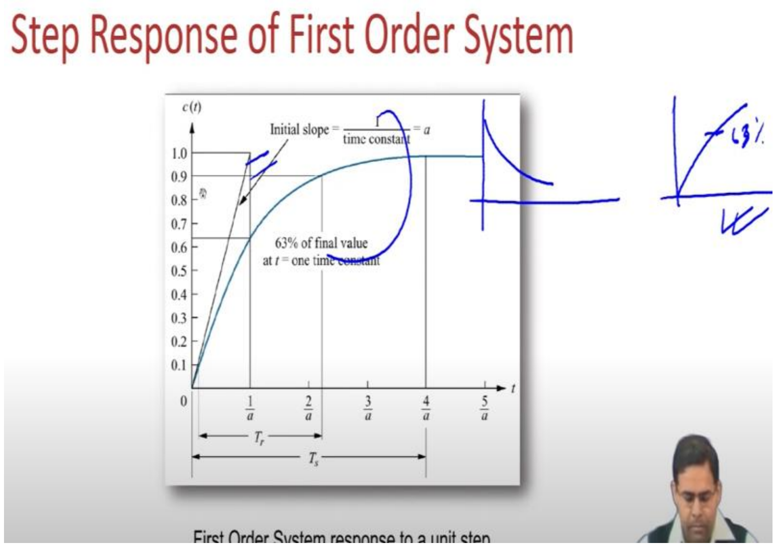

So that is how we have this first order response curve. You see this is the slope of the curve the initial slope, and that is 1 by this time constant. Let us call it a. And we define this to be the time inside which you get to 63% of the final value. There is a good reason for doing that we will see. And then based on this time constant we can define several interesting quantities like rise time and settling time. So let us see. So the time required for the system's output or to decay to 37% of it is initial value in this case, or the other way, in this case to rise to 63% of it is final value is known as the time constant tau. It is nothing but as you can see the reason we talk about the time constant it is basically the inverse of the slope of this line.

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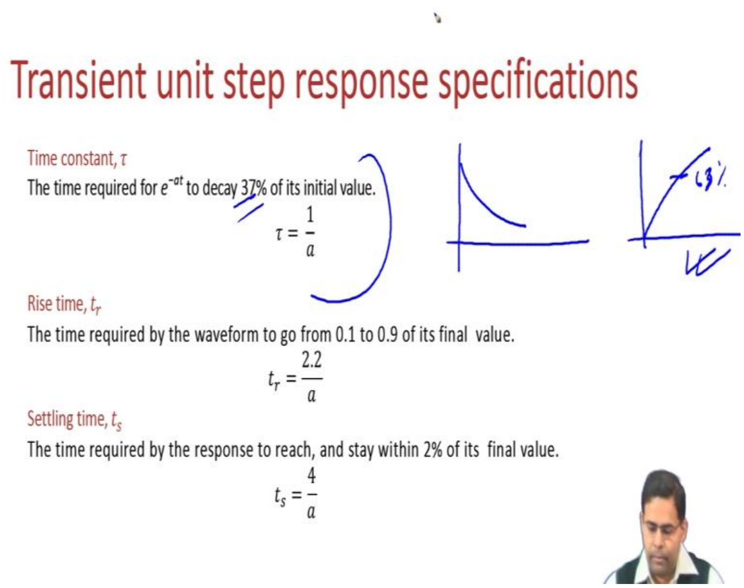

So once we know the slope, we know that how much time, by how much time it will decay to 37% in case of the unit step, or it will increase to 63% in case of the unit impulse. So this is about the unit step. So the time required for it to go from 0.9 to, 0.1 to 0.9. So one simple thing here. So this is when we are talking to decay 37 to decay 37% of it is initial value, that is what is your. This is with respect to the unit impulse response. And when we talk about the unit step, you are coming up to 63% of the initial value.

So this is with respect to the unit impulse here, and this is with respect to the unit step here. In that way when we talk about this, we have 63% of the final value and we are writing in another way that is 37% of it is initial value. So I hope that is kind of clear here. So if we continue from here, then we have the next important thing, which is rise time. Means when the time inside which I get to this zone that is within 90% of the final value.

That is the time required for it to go from this position and it can be figured out that this rise time is nothing but 2.2 by a. So when we are measuring rise time we are talking from, talking about the time interval inside which it moves from point 10% to the 90% this thing. And this is related with the time constant a with inverse of the time constant or the initial slope a, given by this relation that is 2.2 by a.

And the time required to go inside 2% of it is final value, that is what we call as the settling time. This is my settling time we are considering a 2% but in different books you can figure that well the settling time may have a 2% or 10%, different specifications can be there so here if we take the definition with 2%. So it is saying that starting from 0, the time required inside which I have a guarantee that the first order systems response would go to 98% of the final value, or in other words within 2% of it is final value.

So that is how it is and this important fun fact is, all these values can be nicely related to the initial slope. You see the initial slope is a, which is inverse of the time constant and the time constant defines the 63% rise time. So this time constant like we said is 63% of the final value and here we have the rise time which is the time for going to from this 0.1 to 0.9 of the final value and that is related to the initial slope like this.

And this settling time which is 2% of the final value from 0, the time required for that, that is related to the to the time constant in this way. The point is if we can give unit impulse or unit step, in this case is the unit step all these things are with respect to the unit step. So if you give a unit step to this first order system and if you can measure this times you automatically get to know what is the transfer function. Because you can know it is slope and you can figure out what is T and then you are done you have identified the transfer function.

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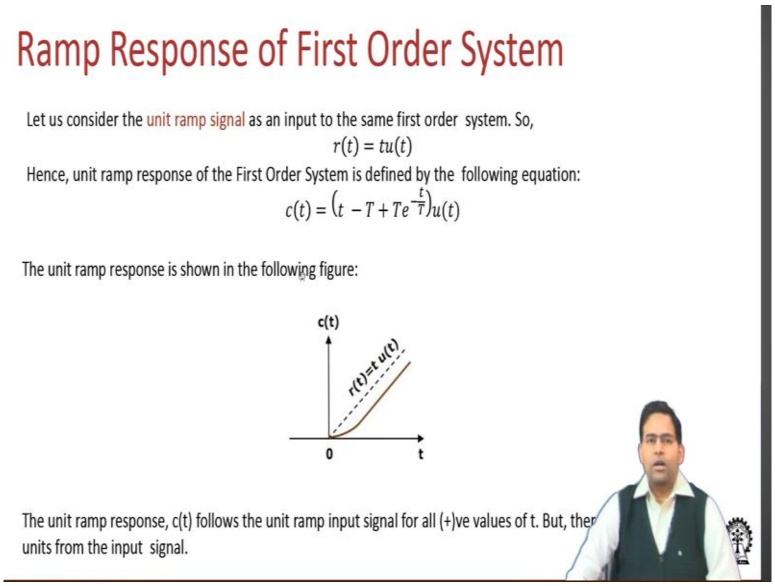

Similarly you can also do a ramp response of the first order system and you can see that well what are the, how the inverse Laplace equation will look like and from that you can actually figure out what is the transient, what is steady state etc.

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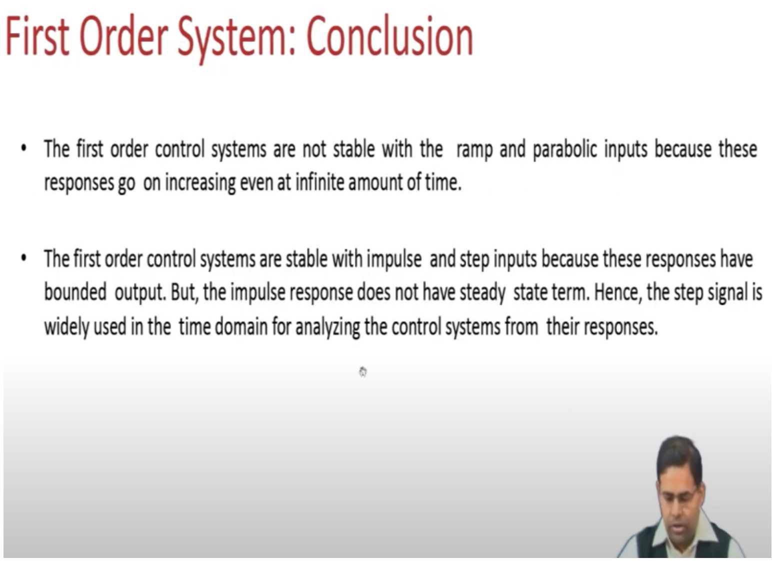

But the thing is first order systems are not stable with the ramp and parabolic input, because you can see how these functions are. We are covering that these things are a bit faster. But it is easy to figure out that well. If this is your input it continuously rises and the system output will also keep on rising to an unbounded value here. This response has gone increasing even at infinite amount of time. And, but they are stable with unit step and impulse.

Because these responses have a bounded output but the impulse does not have the steady state term that we saw. If you do the impulse then it eventually, unit impulse then it was eventually dying down to 0. And hence for this reason the way that you typically study a first order system would be to give the unit step, like we did here. And with unit step we had this well-defined rise time, settling time and time constant and also we had this nice looking curve and we could actually take out these different measurements and we can do a system identification.

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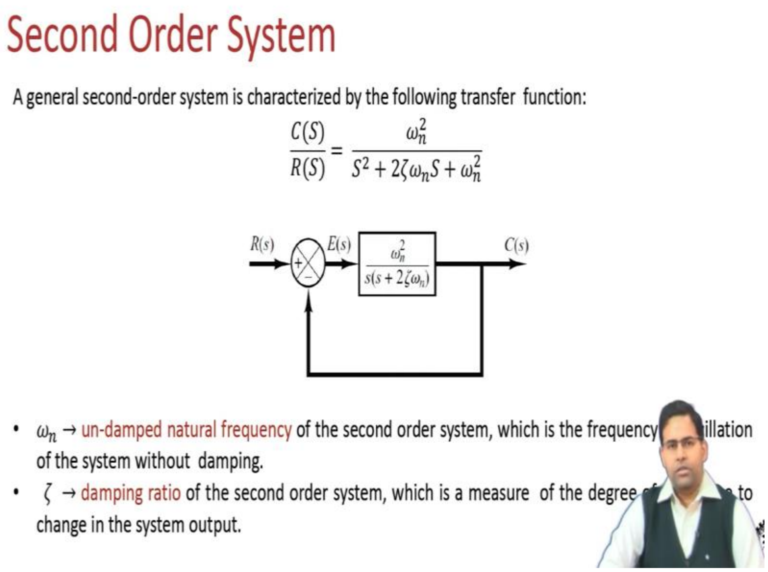

So fine, the next thing that comes is the second order systems and we can see that well for second order system, we will consider the second power of s also in the equation. So typically the second order system will be characterized by this kind of a transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

So you have C(S) by R(S) which is given by this closed loop. So this is the open loop gain here and this is the closed loop function here. And here Omega initial is what we call the undamped natural frequency that is the frequency of oscillation if we do not have damping. And the damping of the system is given by this Zeta. All these things have got their physical meanings. For the space of time we will go a bit fast here but you will get the details about all these things in any basic control theory book.

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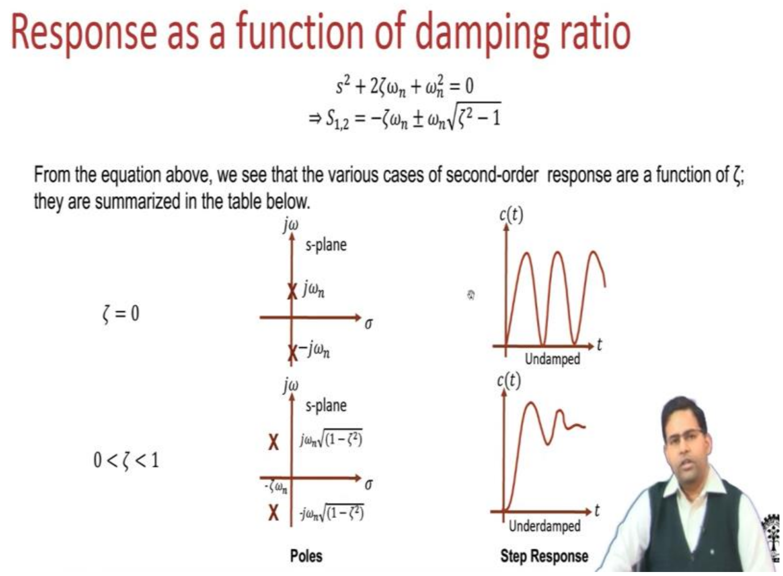

Now what we are interested in is to figure out how the location of poles and zeros vary for second order systems with respect to different possible values of undamped frequency and damp ratio. So, if you see, if you take the roots of the denominator and if you compute the poles if you see the poles are located at this position, - Zeta Omega and plus minus Omega n times Zeta Square -1. So, if you take Zeta as 0 you have imaginary poles a conjugate complex pole pair complex conjugate I mean imaginary pole pair.

If you have Zeta between 1 and 0 you have the complex conjugate poles which are located on the left half hand plane. And if you have damping is 0 you will see that the system output is undamped with this natural frequency that is Omega n. That is why this term so that is why I say this is an under this term is known as the undamped natural frequency because it dictates the frequency of oscillations here.

And if you have a damping value, then the damping is less than 1 you have an under-damp response which will look like this and this is an undamped response. Because it is fully oscillating and the oscillation amplitudes are not decreasing. Of course, all these are step responses and if you have Zeta = 1 then the imaginary part vanishes and what you have is 2 poles located at the same position in the left half plane that is - Zeta Omega n so this is the response here.

And it is the response of a critically damp system as you can understand for a system it is very difficult to set the values like this and get critical damping. But if you have damping to be more than 1, then you have 2 poles on the real axis in the left half plane. And you have the response as an over-damped system. Even this you may not choose to use because this is a slow response. Whereas this is a first response and when you design a control system you will like to have this kind of a response so this will be your target area of operation.

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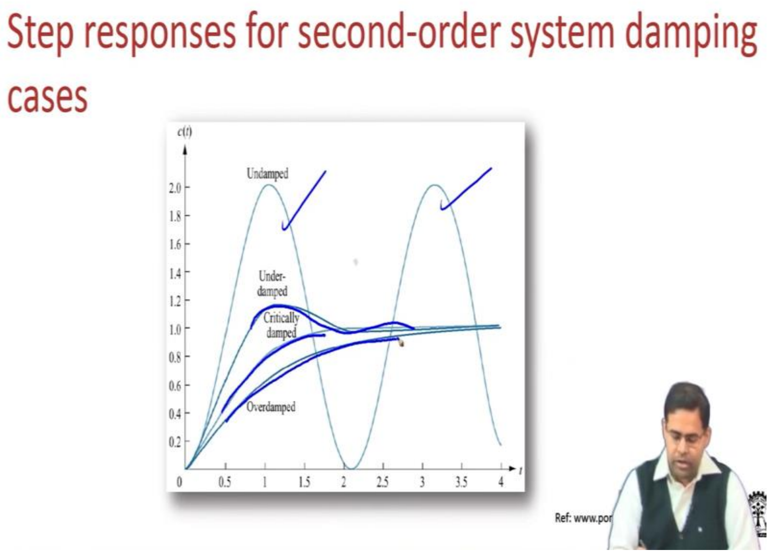

So if you plot all these curves together, here you have undamped, here you have under-damped, here you have critically damped, and here you have over-damped responses. Undamped which you do not want to have because it is like unbounded oscillations. And under-damped, it is rising quickly and it is settling down. Critically damped, something that you want but is difficult to attain. Over-damped, there is no oscillation, but the time it is reaching the settling time it may be, so it is not rising as fast as the under-damped situation.

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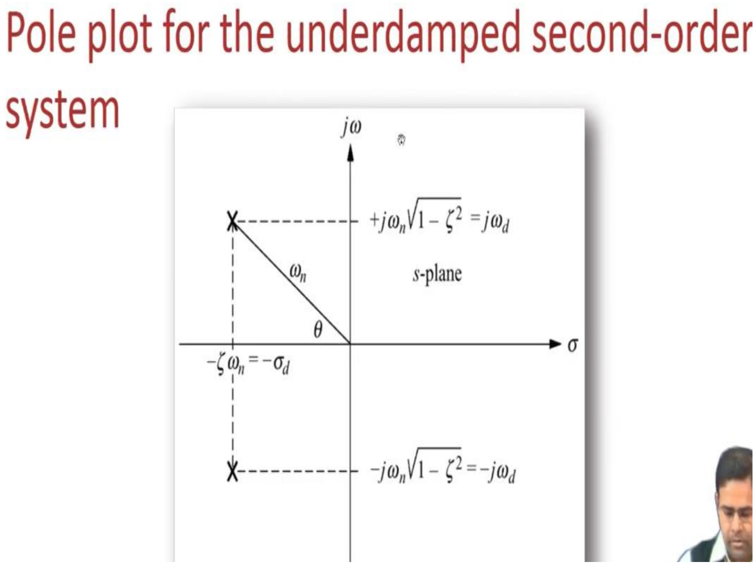

So see like we have discussed that we will like to have the under-damped situation because the response is fast and also eventually it becomes stable and unlike the other cases critically damped difficult to attain and over-damped response is slow. So, if I want to be in the under-damped zone of my design. So what we will have is something like this. We will have the Zeta value between 0 and 1 and we will have complex conjugate poles of the system which will be located at here and here. So for them, the real part is minus Zeta Omega n and the imaginary part is plus or minus j Omega n times square root of 1- Zeta Square. So maybe we will stop here and in the next lecture we will start from here. Thanks for your attention.