Foundations of Cyber Physical Systems

Prof. Soumyajit Dey

Department of Computer Science and Engineering

Indian Institute of Technology, Kharagpur

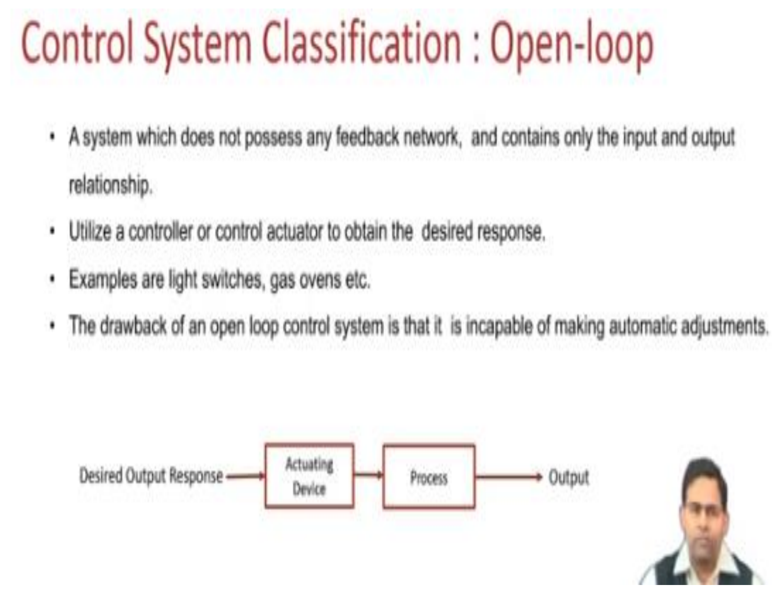
Module No # 04

Lecture No # 20

Dynamical System Modeling, Stability, Controller Design

Hello and welcome back to this lecture series on Foundations of Cyber Physical Systems. So, I believe in our last lecture we have been introducing briefly these concepts of mathematical control theory and so we will just continue from there. So, we talked about what are the goals of control systems in general. And the next thing we will talk about is how are they classified.

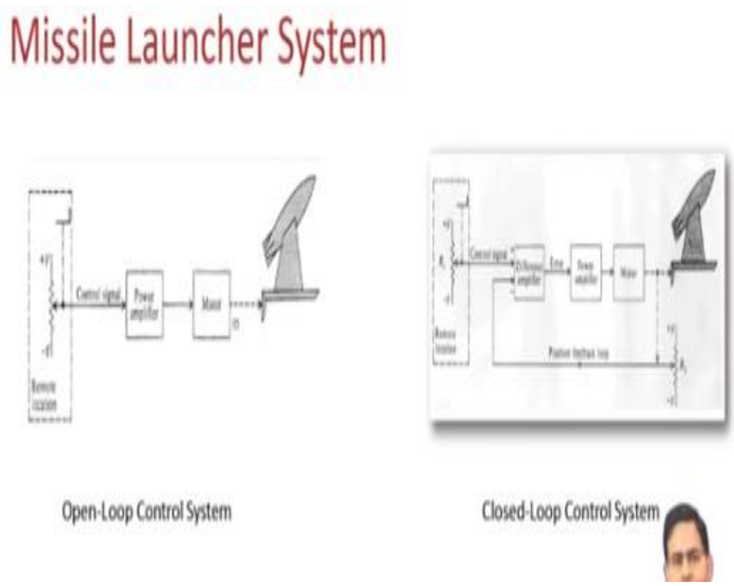
(Refer Slide Time: 00:49)



So a control system can be an open loop system. So that means there is no feedback we talked about feedback, but well there can be control systems where there are no feedbacks. That means there is an initial command right and based on that, I mean, there is an initial command which has been decided based on that well what is the output response I want, and accordingly, the command has been decided and this it goes to the actuating device, and, this actuating device connects to the physical plant or process and then we get to see the output, right.

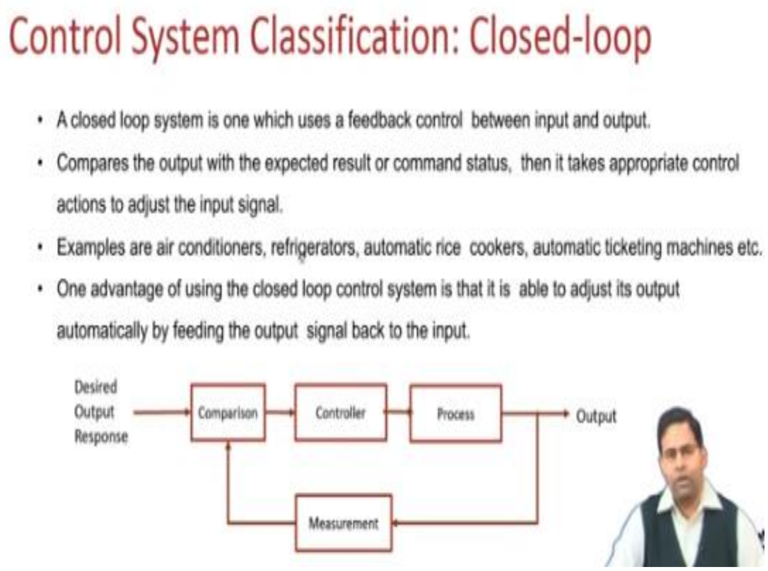
So, for example you take a light switch or a gas oven etc., so you have a command and you can see the output. Well some of these responses may be debatable. Like whether well I can see how the light switch output is the illumination and I can maybe control it, I can look at the illumination out of the gas oven and then I can control its amount of, the amount of turn in the knob etc.

(Refer Slide Time: 01:48)



But definitely we will have open loop systems. For example let us say this one. So let us say there is a missile launcher and the way you have designed this is like this. So again, this is taken from this reference page. So let us say there is a control signal based on the and this is in a remote location. So, you cannot really see it. There is no visual feedback also, okay, and from this remote location you decide that well what should be this actuation command and accordingly a power amplifier will send power to a motor and it will just control the angle of rotation here. Let us say it is like that and there is no visual or image-based response, okay. If you have an open loop control system.

(Refer Slide Time: 02:34)

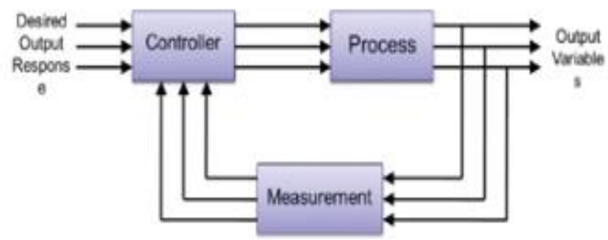


However typically we will like our control systems to have a feedback right the feedback will compare, that well, based on my initial control command what has happened at the output based on that it will take the measurement, it will compare the measurement with some reference input that is there, and again it will adjust the control value and that is how it will keep on happening, keep on executing the control loop in multiple iterations, okay. So, there can be several examples.

For example, the temperature control system in a refrigerator or an air conditioner etc. etc., right. So of course, that is how typically you will like to have a missile launcher system also designed. So, you will like to have this angular command here and then the measurement you will like to see here that will what is the resulting angle, and then you will like to take it back here through a difference amplifier which tells you that while what are the real error in what the angle I want, and what is the actual angle happening and then this power amplifier will actually send a modified power signal to the motor, right. Of course, I mean, I have not done the controller here it should be.

(Refer Slide Time: 03:45)

Control System Classification : MIMO



Multi Input Multi Output (MIMO) System

So it can be for it, I mean, in general this can happen for a complex system where the where it is multi-dimensional. You have a complex multi-dimensional system for which the output is also multi-dimensional and the controller also takes multiple measurements so an x, y these are all vectors right. And based on those measurements and based on multiple references with which you will like to compare those individual measurements. The controller will also give multiple control commands. So, all I am meaning is x, u, y, the reference everything can be a vector and this will lead us to a multiple input multiple output or MIMO system here, okay.

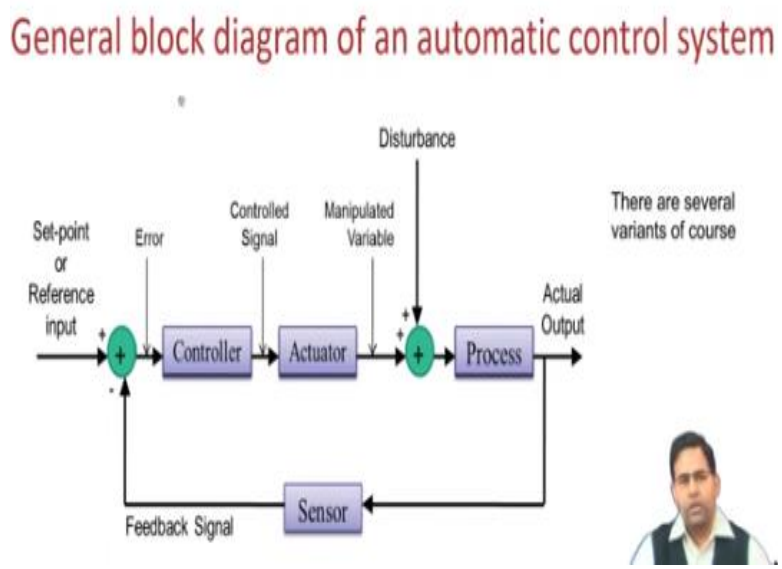
(Refer Slide Time: 04:24)

Control System Components

1. System, plant or process
   - To be controlled
2. Actuators
   - Converts the control signal to a power signal
3. Sensors
   - Provides measurement of the system output
4. Reference input
   - Represents the desired output

So just to sum up we have talked about the system plant to be controlled the actuators which convert the control signal to a real power signal for the system or the sensors which sample the measurements and send them as values to the controller, right. And the desired input which tells that well, what I want my output to really be.

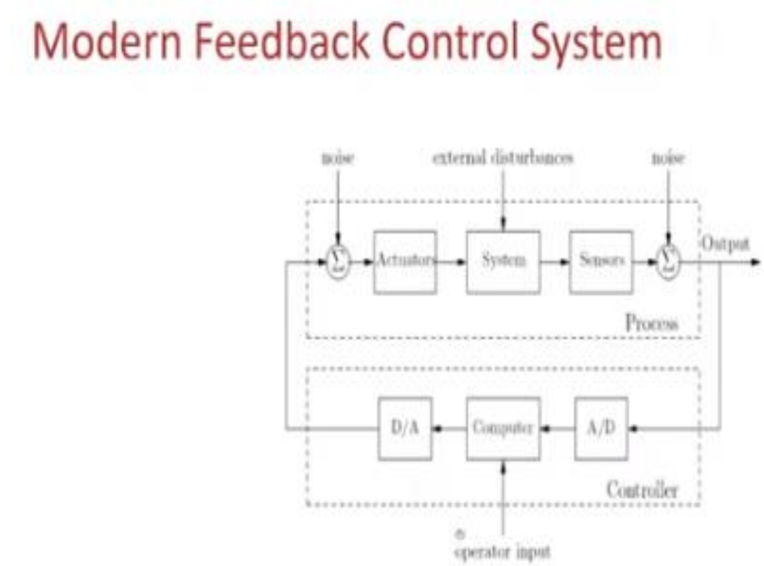
(Refer Slide Time: 04:46)



So based on all these values, I can have this kind of a following block diagram. This is a classic control system block diagram so there is this set input or reference input and you have this feedback signal from the sensors and based on this error your controller is acting and providing a control signal to the actuator. An actuator will actually manipulate the variables and in the process these actuated values can be disturbed.

And then inside the process you can also have some process noise and measurement noise which will add for the disturbance, I mean inside the process and at the output, and then you will get a disturbed measurement, right. So that is our typical control system works and again I will just repeat this is a very basic block diagram in practice there are several variants of them.

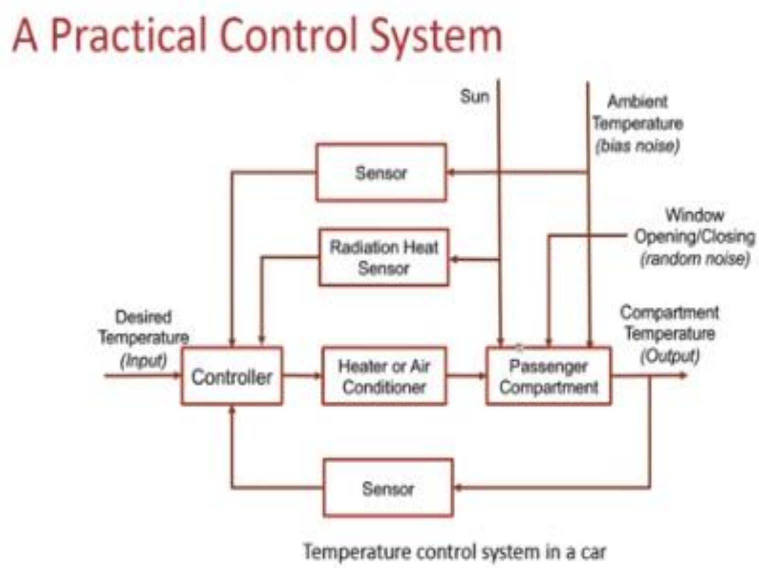
(Refer Slide Time: 05:36)



So now if I look at a modern control system again what will happen is you have the process dynamics the output is sense sampled using sensors and the measurements can also have some noise included. And this output which is an analog output goes to the control computer where at the interface there will be an analog to digital conversion, because the controller may be software executing in a controller in in a computer which only understands these digital inputs only.

And it will generate a digital control output and that would be again converted through an analog signal which in the process of transmission as well as actuation can be disturbed, right. And then the actuator is converted to a power corresponding to signal based on the transducer required. It may be a pressure transducer it may be a hydraulic transducer whatever, it depends on the system that I am controlling and then it is fed back to the system, okay.

(Refer Slide Time: 06:38)



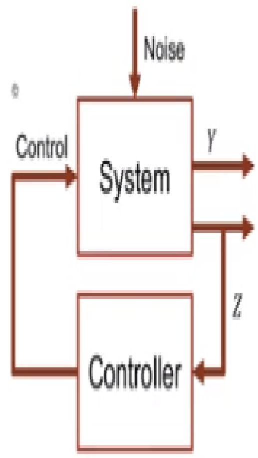
So some simple examples we have that typical controllers what they can do. Of course, these are very rudimentary examples here. We will have controllers for many other complex works also. So, this is an example of a temperature control system. I believe you will find this block diagram from in in the famous book on Modern Control Engineering by Ogata, right. So you have the sun's temperature which is kind of heating up the passengers compartment, right. The ambient temperature is sensed inside the vehicle, right, temperature control system of a car.

And you also have a desired temperature and you have the radiation heat sensors, right. And you have the compartment temperature and based on all this that controller is deciding whether it will, I mean, act as in an air conditioning mode or a heating mode and accordingly it will manipulate the temperature inside the passenger’s compartment, okay.

(Refer Slide Time: 07:36)

State-Space Representation

1. **Input variables:**
   - Manipulative (control)
   - Non-manipulative (noise)
2. **Output variables:** Variables of interest which can be measured or calculated.
3. **State variables:** Minimum set of parameters which entirely summarize the system's status.



Now this is again a few, I mean, a recap, a pictorial recap what we discussed earlier that we arrived at a state space representation and now if we want to write that state space representation inside in with a feedback loop this is what we have. You have the system with input variables which are the manipulated variables, the controls, and the non-manipulative ones which we cannot control which are the noise right. And based on this your system will evolve and produce an output measurement y, right. And you will and best and based on, you can take a subset of this and you can use them for input to your controller.

(Refer Slide Time: 08:18)

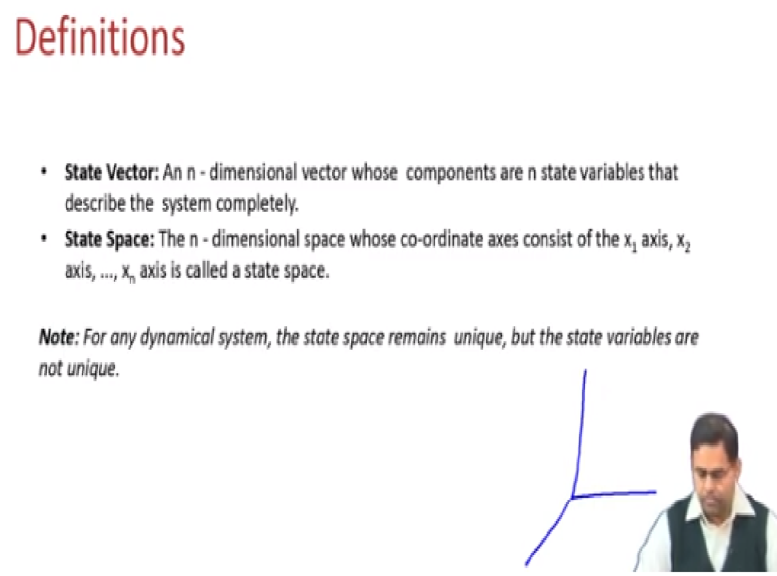
Definitions

- **State:** The state of a dynamic system is the smallest number of variables (called **state variables**) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t = t_0$, completely determine the behavior of the system for any time $t >= t_0$.

Note: *State variables need not be physically measurable or observable quantities. This gives extra flexibility.*

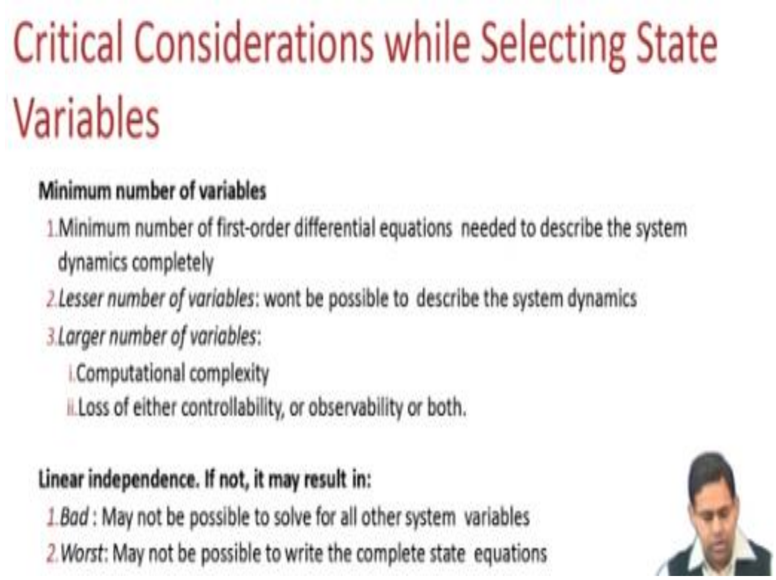
So, like we have already discussed what is the state variable, we will just recap that in a formal way. So, the state of a dynamical system is the smallest number of variables such that the knowledge of these variables known at certain time point together with the knowledge of the system flow function, right, we will completely help me to, will help me to determine the complete behavior of the system at any time in future, right.

(Refer Slide Time: 08:46)



So based on this like we discussed that you have you can consider create an n-dimensional state vector, and these vectors the possible set of values, these vectors can take is what gives you the state space. So, for example if you have three dimensional system then you have variables x_1, x_2 and x_3 and the given the systems trajectory restrictions and function it can take set of values in this state space, right.

(Refer Slide Time: 09:22)



Now this is a point we will try to come back again. We are always talking about minimum number of variables. That means we need to be aware that what is the functionality that we are targeting. A system like a vehicle may have many functionalities. But if we are talking about adaptive cruise control, we are interested in only those parameters of the vehicle which affect the velocity, right.

So, using that idea we can identify what is the minimum number of first order equations that we need to describe the system dynamics completely. And if so, that if we now leave out any one of those equations and the corresponding variables which the equation updates, then we will be missing some part of the system dynamics, okay. So, based on that we can identify the set of state variables. And why do I want to have this minimum set? The idea is if I have extra variables which do not relate to the dynamics of my interest, then I will have more things to compute right, which I do not want to do in a real time setting, okay.

And also if we have extra variables for which the relations are not well defined, then maybe, I mean, I may not be able to evaluate the systems value for those variables, if I do not take into consideration all the variables which actually affect such things. Now these are things we have already discussed.

(Refer Slide Time: 10:50)

State Variable Selection

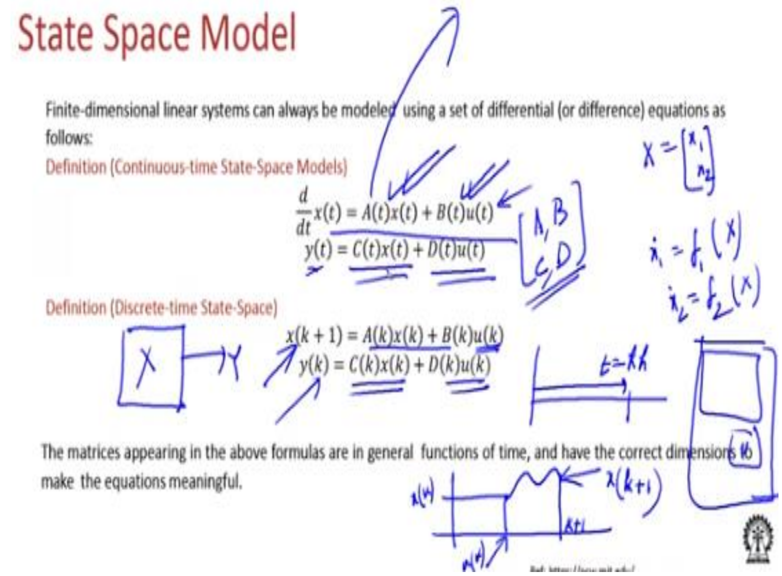
- Typically, the number of state variables (i.e. the order of the system) is equal to the number of independent energy storage elements. However, there are exceptions!

- Is there a restriction on the selection of the state variables ?
  YES! All state variables should be linearly independent and they must collectively describe the system completely.

Now let us look into I mean how the state variables can give me the set of equations and what is the standard form in which one would really like to write those equations. And this is an important point that the state variables should be linearly independent and they must completely describe the system. Now what do I mean by independence here? It means that well, I mean, I should not have a variable in my system of equation, so that its value can always be inferred from the equations of the other variables.

Then the equation which is giving me the evolution of this variable is not really necessary, right. So that is why the state variables all of them should be linearly independent and so that is an important point. And otherwise it also will mean like we said that if I do not have independent variables, right, if I have more variables than equations then it may, I may not be able to come up with solutions to them. And also, if I have more number of equations in the variables then I may not have unique solution at each time point, right.

(Refer Slide Time: 12:01)



So that is the point so we need to have the minimum number of variables which distinctly define the dynamics of the system under question, right. We do not have extra variables which are not related to the dynamics and instead we just have the same number of variables as we have the equations to have unique solutions to all the control to these State variables at specific time points. So typically, that is how you will model the system here so now if we talk about the creating the models.

So, we have already seen that let us say for each vector component we would like to have a functional form, right. So, if we just recap that this is what we discussed, that let us say x was having x_1 and x_2. Then for x_1, let us say you have a function f and for x_2 its flow you have a function f_2, right. So, using all these relations overall, I mean, for the entire vector x, let us say you are able to write something like this. So, this equation is telling you how the state vector x is varying over time with respect to x as well as the control input u, right.

And similarly let us say, we also are able to write a set of equations for the set of observable measurements y. And which tell me that well how y varies with respect to x and the control input, right(13:35). So we have already discussed right that how to construct this form x dot equal to f(x). Now we are saying that well x, I mean, when you have the controller so when I just talk about this x dot equal to f(x) I am talking about all the variables together, right.

But now we have brought in the controller u, right. And we are considering them together as a system right. So, in this case we will say that well this is a control system, so the evolution of x does not only depend on a, but also it depends on this control inputs, right. So, if we write the equations which tell me that how x evolves with respect to its own current value as well as the control input, we have this equation.

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

And then, if we start saying that well how the output measurement changes with respect to current x and control input then we have this equation. And from these two equations we can understand that well, the entire dynamics of the system the plant and the control, I mean, the entire trajectory of the system x and the measurements of the system are defined uniquely, if we are able to compute these A, B, C, D matrices for the system.

Now if you see here when we are writing like this we are assuming a linear system because we have this linear form x dot = Ax, okay. And since we are making A, B, C, D all as functions of time that means this is a time variant system. If it is a time invariant system then you will stop writing A, B, C, D as a function of time here. Now if we write it in a discrete state space we like we discussed earlier, then instead of making x as a function of real time t, we will write both of these equations in this difference equation form.

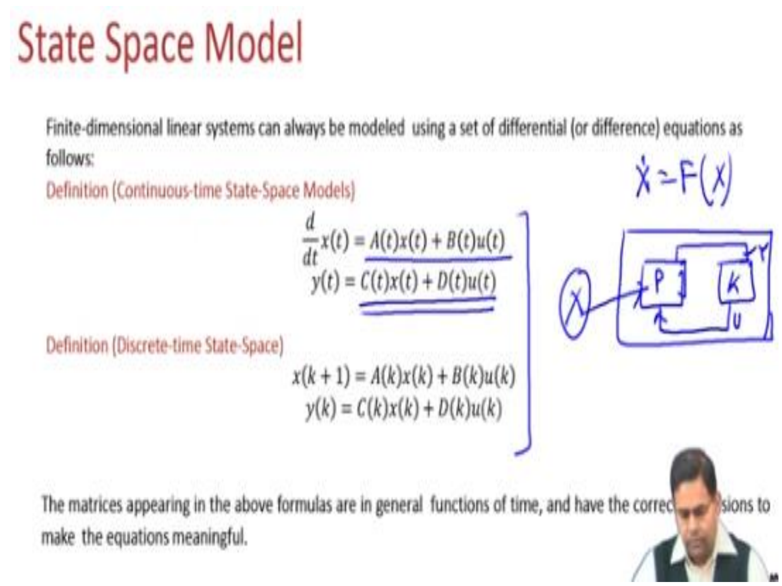
$$x(k+1) = A(k)x(k) + B(k)u(k)$$

$$y(k) = C(k)x(k) + D(k)u(k)$$

So that suppose that the kth time step so this is the k h time step so t = kth times the periodicity h. What is the value of Ax is completely determined by the value of x at k and the control input at k based on them the system evolves in some way. So let us say this was my x(k) and then at this point we give some u(k) and based on that the system evolved like this. And then, at time k +1, this is my x(k +1) value, and x(k +1) we can be computed like this.

Similarly for the system it has this set of State variables x, but not all of them may be observable. What we can observe is some y, and y can be computed as a function of both x and u. Well for most systems we will see that y is some Cx but it may also be a function of u right. So in a general case we can say that what we observe is nothing but some C times x + some D times u here, right. So, if you have x as some n size vector, your A is an n cross n one, right and the corresponding all the other are the other matrices should have suitable dimensions so that the entire thing matches together. Now considering that the system is linear time invariant this will not be functions of time, this A, B, C, D matrices would not be functions of time right. So just to recap again, we talked about how to write the system in this form.

(Refer Slide Time: 18:17)

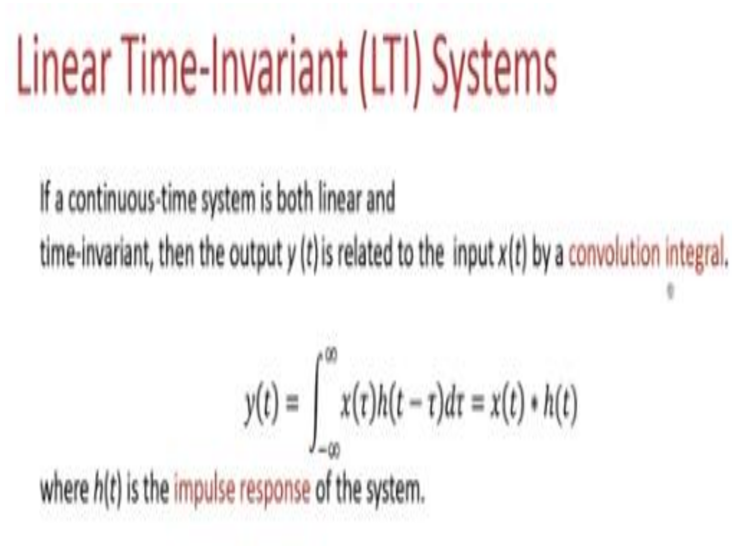


And then we said that well you have to account for the control input itself. So together inside your system if you have the plant P and the controller k let us say, then your plant state x, the plant state x not only changes with respect to the plant's dynamics but it must also have a component where you take into effect the contribution that comes from u. So that is why you will have this form Ax +Bu and then for the output measurement also you have this form Cx + Du, which shows that will how the measurement is affected by the plants state and the control input.

So, x is the plant state u is the control input and this set of equations kind of tell you how the plant dynamics really changes over time and considering time invariant this won’t be a function of time.

Now let us also look into another interesting property of time invariant systems which is this that of course this is we have made a linear assumption here, right.

(Refer Slide Time: 18:38)



And we are able to write this only for the finite dimensional linear systems. And if I have this continuous time system which is linear and time invariant, then the output of this system y is related to the input x by what we call as a convolution integral, which is given by this.

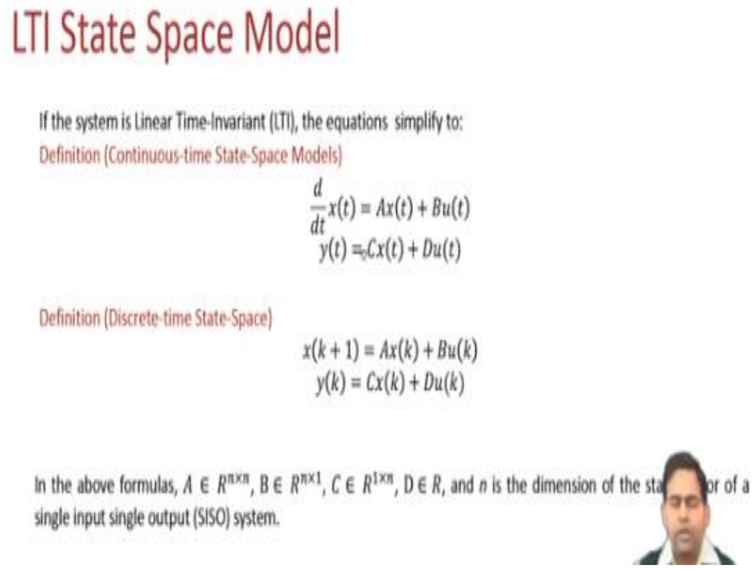
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

As essentially as you can see here, first thing is let us understand what is h. h is known as the impulse response of the system. So, if you have a unit impulse function, right.

What is the impulse response if there is h then what really is going on in the convolution integral. So essentially, you can think that well you are multiplying these different instantiated values of x with h where h is shifted by an amount tau and we were considering this for different possible values of all possible different values of shift. So, it is like how one function is passing through the other function in time continuum, right, and leaving out its effect, okay.

So, I mean if you can study about convolution integral and stuff in any good book on basic controls signals and systems and stuff, right. So, we are not continuing on that here. But just to tell you that this is the property of LTI systems only here.

(Refer Slide Time: 20:03)



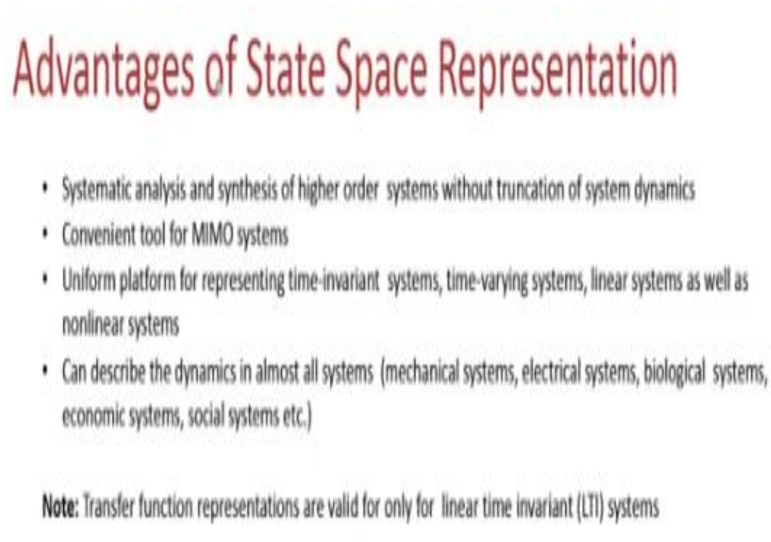
Now for LTI systems if we create the state space model what we will have is like I said that the A, B, C, Ds are fixed right, they are not a function of time. So just going back on our earlier discussion on LTI system which we said that when it is time invariant and when it is time variant and we also said that well it is time variant, then it is also a non-autonomous system. So here we say that well, this is a non-autonomous system because with time the definition of the system is also changing.

And here I mean because time is an independent variable that is here, so you can see that here we have this form where the definition of the system is changing with time right. So here if we try to think of A, what is A? A is basically the time the time variable T itself right in an identity matrix times time in that format right so that is why this was a non LTI system. So, coming to this when the system is time invariant then it will be written in the state space form in continuous time like this.

So, you can see that the function of time has vanished for A, B, C and D and again in the difference equation for the discrete time state phase it will be written like this right. So here if you have an n dimensional system, n is the dimension of the state vector so A is n cross n, B is n cross 1, C is 1 cross n, right, because you have to match the dimensions with the vector x, right. And D is again depending on this control if the control is having a multi-dimensional control inputs then accordingly D has to be matched into that, okay.

And if it is single input single output then as you can understand that well, I mean, u is a single output system so in that case D would be a scalar, right. But in general if it is in a matrix format then well it's not so. D can have my multiple more than I mean one dimension here.

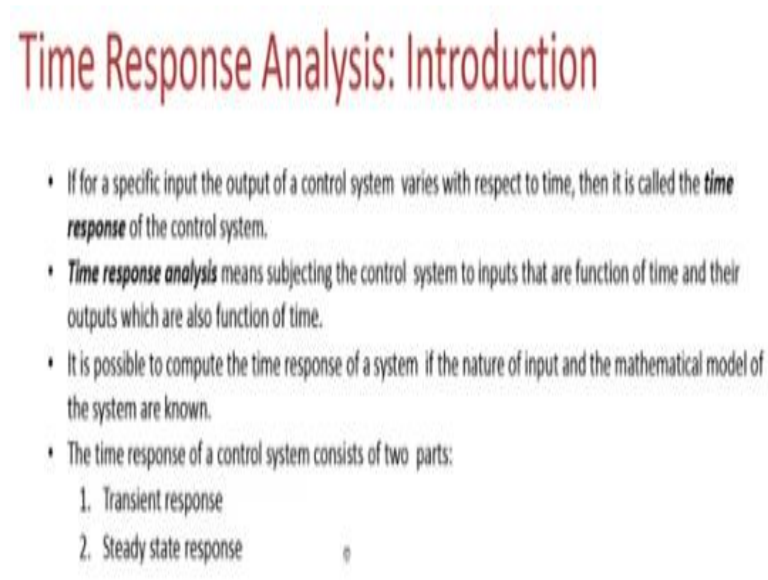
(Refer Slide Time: 22:13)



So this state space representation provides you with a systematic method for analyzing high order systems. And specifically for MIMO systems this would be very advantageous and we will do all our analysis considering the state space form only. And it is able to describe dynamics for most class of systems in general, mechanical, electrical, biological systems etc. And of course, you know that the other popular method so, and also in state space system you can capture a time invariant, time varying systems, linear systems, all of them, right.

I mean it is a uniform platform you can represent time invariant systems, time varying systems. You can also represent linear systems specifically in this format when you have A, B, C, D. But of course, when you have non-linear systems well something else some other factors will also come in right but the state space model is very generic here. And the other popular control system modeling that we know which is the of transfer function-based systems, they are only valid for LTI systems that is the linear time management class. So in general because of the conveniences associated with state space we will be continuing to use this state space representation only.

(Refer Slide Time: 23:36)

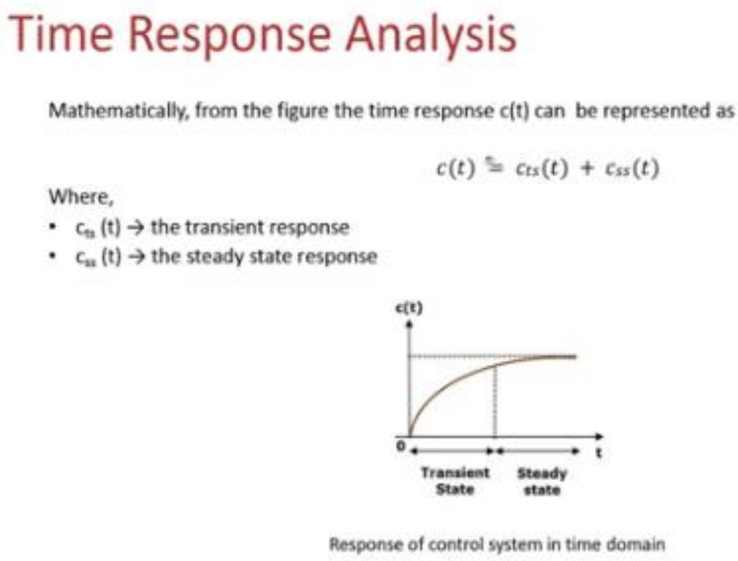


Now the next thing that we will be talking about is analysis of time response right. So let us understand what it means? So like there is some well-known control inputs with respect to generic system design theory and for such inputs how the system reacts and what is the output that we can observe from them, we can actually kind of deduce the system characteristics, I mean, which is basically a method of system identification.

So we will do this study of time response analysis which means subjecting the control system to inputs that are functions of time and we will like to see that well how the outputs are changing with respect to that. And in that way, we will see that well we will try to understand from this response that how the system is really what can be an underlying mathematical model of the system.

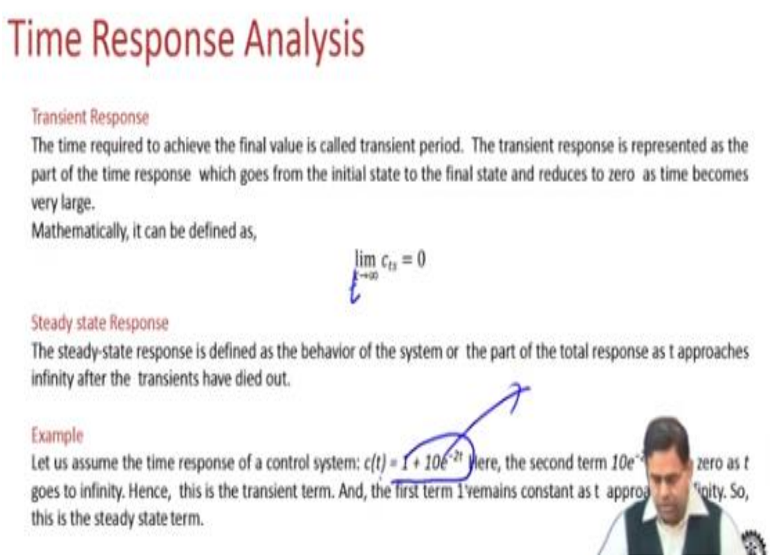
And to go deeper into that, we will first understand this classification that at any time response of such a control system will have 2 parts. One is the transient response right, another is that steady state response. So let us understand what it means.

(Refer Slide Time: 24:48)



So mathematically if you write systems response as some functions c(t). So t is the time there are 2 parts one is the transient and other is the steady state response. so steady state response means with time leading to infinity what is I mean what is I mean what eventually the response becomes.

(Refer Slide Time: 25:13)



So the part of this response. So if I am able to write this response as a mathematical function of time. Like I said that there are 2 parts. Which is a transient and the steady state. The part that eventually dies out is the transient response. That means it is the part which happens initially in the system and eventually it will reduce to 0. So as t tends to infinity, this will die down. And the steady state response is the part that remains with this t term tending to infinity.

For example, let us say you have a control system whose output is defined like this.

$$c(t) = 1 + 10e^{-2t}$$

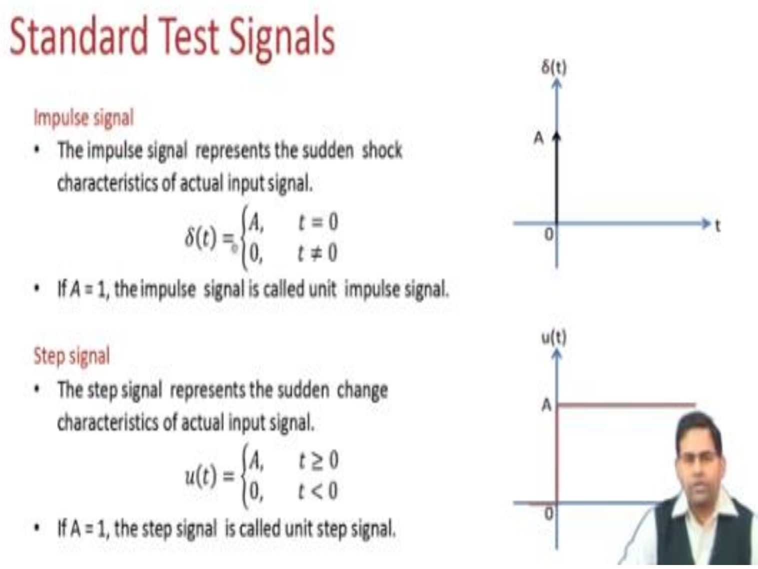
Now if you see this second term is going to die down with t tending to infinity so this is the transient part and this is what will remain. So that is the, so with t tending to infinity you will have the systems response approaching one and that would be the steady state response.

(Refer Slide Time: 26:17)

Now like I have been saying that for system identification purpose there exists some standard test signals, right. And with respect to which we judge I mean we know that well from our basic underlying mathematics that given these standard state signals like the impulse response, impulse input, step input or a constant. I mean, this kind of inputs, I mean if we can observe the system's output and if we can analyze the output function, we are able to kind of infer certain properties of the system.

So, with respect to those we will like to give certain we would like to study this thing that how to identify certain system properties with respect to this kind of standard test signals. For example, if I take an actual system like a vehicle for it a constant velocity or certain acceleration this can be standard test signals. But in general when we speak in terms of state variables, there this impulse they are the step these are the are the standard I mean standard variables. And if we interpret them for let us say in an automotive case, they would be in terms of velocity or acceleration.

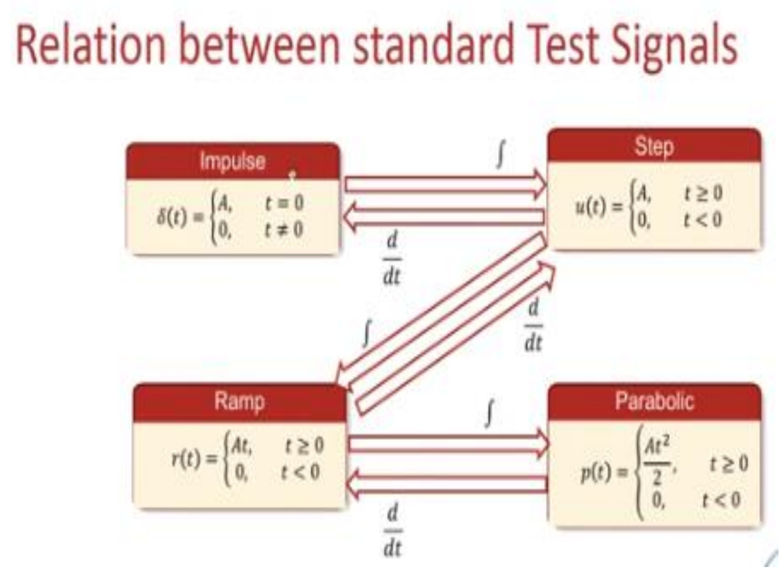
(Refer Slide Time: 27:44)



So let us see here when we talk about standard test signals. So we know that one of them is the impulse signal. So it is a signal that is defined. So let us say you have a signal which has amplitude A at time equal to 0 and before and after that there is no value. So that is an impulse with an equal to 1 you have an unit impulse. Similarly you can have a unit step or an S step. So a step signal is like this so it has an amplitude A from time equal to 0 plus.

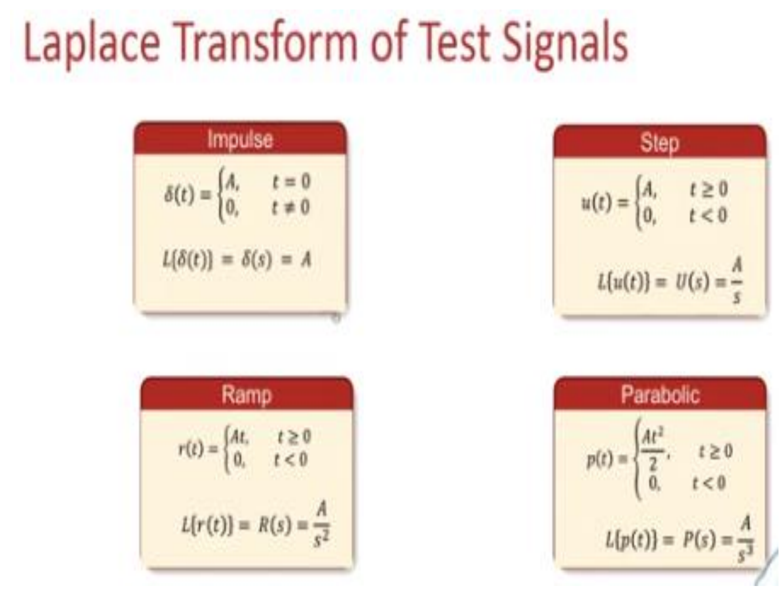
And if A is 1 then what you have is a unit step. So these are some examples and for that if I just interpret them with respect to a vehicle model you are talking about constant velocity constant acceleration this kind of signals. And similarly just like unit step and unit impulse, you can have a ram signal right. So if you can have a ramp like this right. And also you can have a parabolic signal, so it will have this form, At square by 2, so if that represents a parabola from the origin right. So similarly, you can have a unit ramp or a unit parabola based on selection of the amplitude A of the signal.

(Refer Slide Time: 29:00)



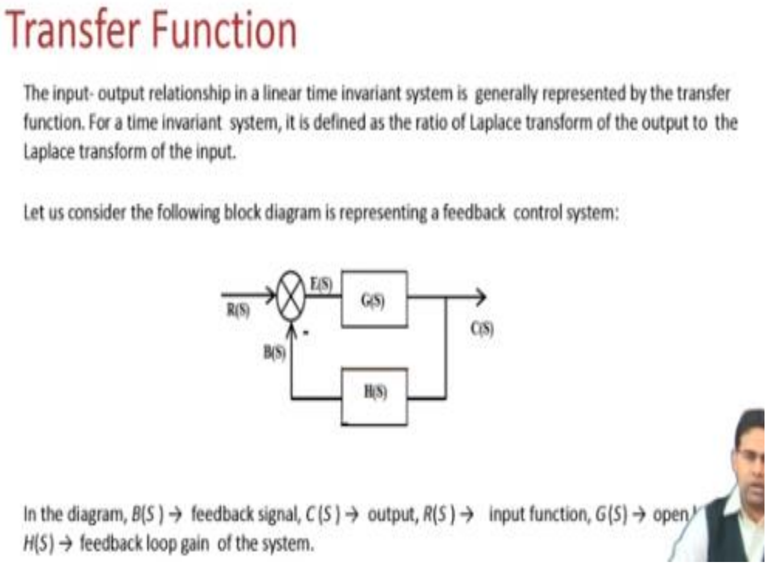
Now there is a nice set of inter relations between these test signals so if you keep on integrating then from impulse you get step, from step you get ramp, and from ramp you get parabola. And the differentiations from parabola will give you the ramp, the step, and the unit impulse signal.

(Refer Slide Time: 29:22)



Now assuming that we all have some basic knowledge in Laplace transforms these are the well-known transforms that we have. So unit impulse will give you one. And otherwise you get the amplitude for step you have this nice form, A by s. Ramp is A by s square and for parabola you have A by s cube, right.

(Refer Slide Time: 29:40)



And so what we are doing is we are trying to now see that we are working under the LTI assumption. And we are now looking at these different this control diagram in the form of this transfer, in this transfer function in the Laplace domain view, okay. So you have this input output relationship in a linear time variant system and it is represented by this transfer function. And for a time invariant system as we know, it is the ratio of the Laplace transform of the output to the Laplace transform of the input which is the transfer function. And from that we have this general block diagram representation. So maybe with this we will end the current lecture and we will again resume from this point. Thank you.