Statistical Learning for Reliability Analysis
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Lecture - 10
Discrete Probability Distribution (Part 2)

So, hello once again so in continuation to our earlier lecture on discrete probability distribution, where we have discussed binomial distribution, uniform distribution, binomial distribution, multinomial, and hypergeometric distribution.

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In today's lecture, we will be discussing few other discrete probability distribution. So what are those we will be discussing negative binomial distribution, geometric distribution, Poisson distribution. And also finally we will discuss the descriptive measures of all these distributions.

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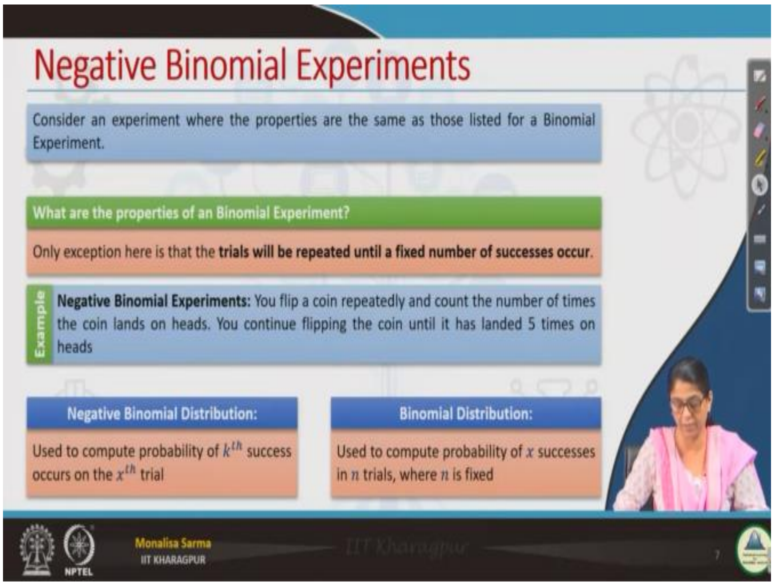

So, first is what is negative binomial distribution? In negative binomial distribution if you guys can remember binomial distribution what was that our number of trials are fixed that was n we have specified it by small letter n remember that was the number of trials or number of trials are fixed in this number of trials, we our random variable was the number of successes.

So, variable is something which changes so, we have taken the random variable as the number of successes. So in n trial, whether the number of success is 0 success, 1 success, and it can go for X can take value from 0 to n, that was our random variable is the number of successes that was changing, but variance in some other experiment, but we get to see is that where the trials are repeated till we get a fixed number of success, meaning here.

Now number of trials is not fixed that is number trials is not n the number of tails can be anything, but we are interested in getting a fixed number of successes that means suppose we are interested in getting total 5 success. So, our number of successes cannot be a random variable. So number of success is fixed here, is not it? It cannot be random variable now what is the random variable here, our random variable is the number of trials.

So now my X is number of trials, rather number of successes, number of success is fixed that is the negative binomial distribution, that is the only difference between binomial and negative binomial, there is not a very small difference, but it is a big difference, but that is the only difference between binomial and negative binomial distributions. So, negative binomial distribution and example you flip a coin repeatedly and count the number of times the coin lands on head, you continue flip the flipping the coin until it has landed 5 times on the head, there continues my interest is I should get 5 heads.

So I am continually prepping it till I get 5 heads when I got 5 heads I stopped it. So, my number of trials is how many trials I needed to get 5 heads so that is my random variable. So, negative binomial distribution used to compute probability of the kth success occurs on the xth trial k is fixed x is a random variable, whereas in binomial distribution we use to compute probability of x success in n trial, where n is fixed. So, that is the only difference between the negative binomial and binomial distribution.

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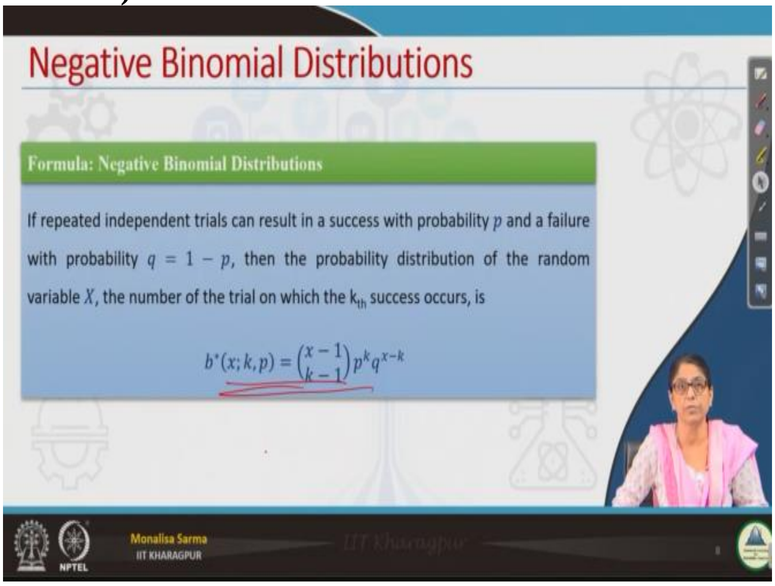

So, similarly, our definitely if this is the only difference. So our formula for calculating fx will also be same almost same the only difference here you will see since we are interested in finding out kth success depends on my last trial, last trial will be the k number of success. So trial before to that, if my last trial is x, trial before to that, that means an x - 1 in x - 1 I will get k - 1 success this can be in any pattern as we have seen in binomial distribution.

How it can be in different pattern success success, failure, failure, failure, failure, success success I have shown 1 example if you can remember, if you do not remember please go again go back to the lecture again, you will see that I have worked in a blue board showing you how we have done so similarly here, that means on the xth trial, I got my kth success. That means the all the trials before these xth, before this point is the x - 1 and this x - 1 trials, how many success I got? I got k - 1 success it can be in any order.

So that is why it is x - 1 C k - 1. And total how many successes are there? Definitely I am looking for k success. So total that will be k p^k. And how many non-success that is 1 - p that is q, q is x - q, $q^{x - q}$. This is the formula for binomial distribution negative binomial distribution.

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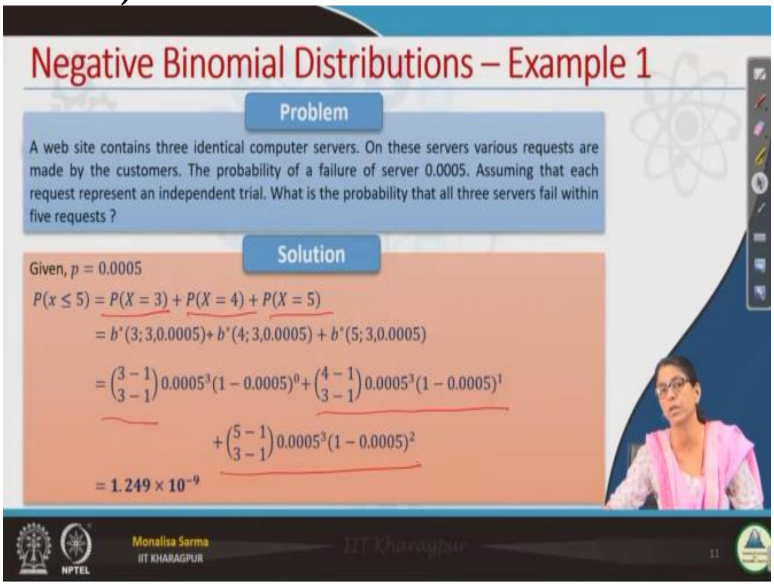

So we will see an example a website contains 3 identical computer servers on this servers various requests are made by the customer the probability of a failure of server is 0.0005. The probability of there is a probability failure assuming that each request represents an independent trial here also trials are independent like what we have seen in binomial distribution, what is the probability that all 3 servers fails within 5 requests?

See an interesting question, but we have to find out? We have to find a probability that all 3 servers fail within 5 requests. So, what is this means? Our what to say we are interested in finding out, our number of failures are fixed, what is the number of failures? Number of failures is a random variable here. So, we are interested in finding out x is that is my, what to say my number of failures.

It is what it is given one thing assuming that each request representative, what is the probability that all 3 servers fail within 5 requests? So, within before 5 my all 3 servers would fail before or on 5. So that means I am in my number of failures are fixed that is 3 failures and how many trials maximum my trials will be 5. So, x is the so, my 3 servers may fail in that third request.

My third failure I may get in a third request my third failure I may get in a 4th request? My third failure I am getting a 5th request whenever you get the third failure stop it. So my question was what is the probability that all 3 servers fail within 5 requests? So it has to fail within 5 requests within 5 requests means it may fail within 3rd requests it may fail within 4th requests in may fail within 5 requests, is not it?

It definitely it cannot be less than 3 because then 1 request there can be at the most 1 failure. So, it might my third failure may be in the third requests. So that is p (x) = 3, my third failure maybe in the 4th request my third failure maybe in the 5th request. So x is the number of request that is the random variable. So, what is the formula what we have seen, so, this putting it in the formula, that is all nothing else, just if we put it there, how do you put it in a formula if we might 3rd request in a 3rd fail, so, that is x - 1 / k - 1.

So x - 1 / k - 1 x p^k 1 - p how much? k - x so, here it is 0. Similarly, my third failure in the 4th request, this is in the 5th request, so that is negative binomial distribution.

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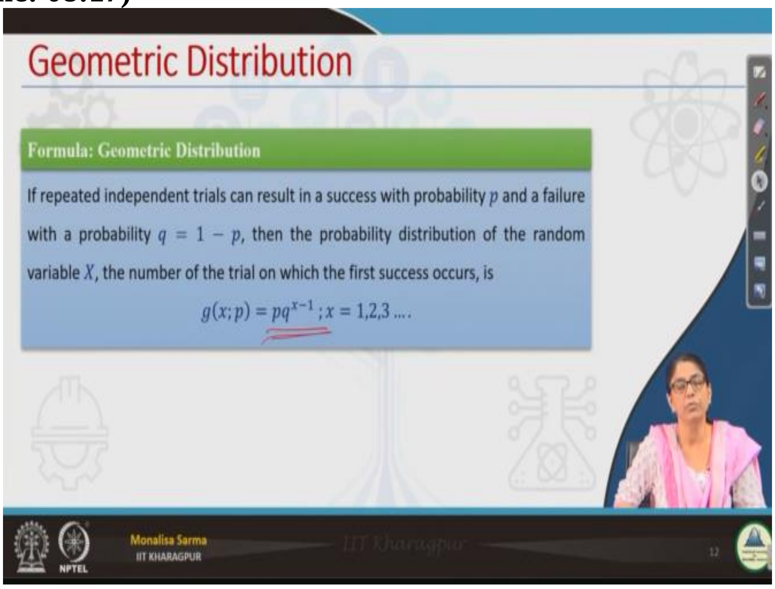

So, now, this another type of distribution which is also in the same angle, same direction, I should not say angle same direction that is geometric distribution in geometric distribution, what we are interested? We are interested in finding out the number of trials we need to get the first success. Here also, my success is fixed success is, 1 just 1 in negative binomial my success can be anything but the success was fixed.

Here also my success is fixed and that is only 1. I am interested in finding out how many trials do I need to get the first success? The applicability, first let me tell you the applicability of this type of geometric distribution. Suppose, you are installing some company is installing some telephone and connectivity. And maybe we are interested in finding out when we dial in number after how many dial we get the how the line gets connected.

For dialling how many times on my line gets connected? I have tried it once the line did not connect, get connected in a busy time maybe I have tried it again it did not get connected, I tried it again it did not get connected. So suppose my line got connected on my 5th trial. That means the design there has to be something some problem with the design, no one will try wait for 5 times dialling and then getting the connection.

So, this sort of thing the geometric distribution is essential based on this people can find out so that there is problem in the line and accordingly steps can be taken. So there are many such cases where geometric distribution has great applicability. So my first success, here also the random variable is the number of trials remember always the random variable is that, that changes.

So, in any problem first you need to find out what is the random variable if you know and if you find out what is a random variable, then finding the probability will not be different than you will have to then finding out which it falls in which broader distribution, whether it falls in binomial, multinomial, hypergeometric, negative binomial, geometry what Poisson and whatever it is, first find out what is the random variable.

So, here random variable is the number of trials, and we were interested in number of trials we get the first success. So, the formula for this is quite simple, it is simple p (q) so, there are total x trials, is not it? So, in x trial, we will get 1 successor the last one is success, is not it? So, 1 success means that is p and in rest x - 1 that is the failure and failure is not distinguishable.

So, you do not have to do any convenience or something like that. So, it will be so, simple q to depart x - 1. So, simple this is the formula p into q x - 1. So, when x can take any value from 1, 2, 3 like means, we can get the first success in the first trial itself we can get the first x in the second trial. So, x can take anywhere from 1 to 3 so, that is geometric distribution.

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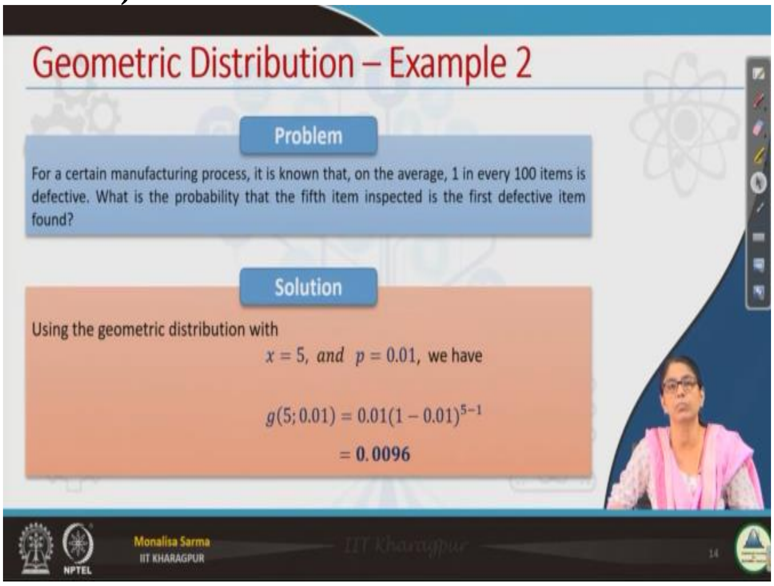

So, problem for a certain manufacturing process it is known that on average 1 in every 100 item is defective what is that 1 in every 100 item is defective that means this question this particular statement what it gives? This particular statement gives us the probability of defective items what is the probability of defective items? 1 / 100 is the relative proportion called the defective item is 1 / 100.

So, what is the probability that a 5th item inspected is the first defective item found, this this reminds me we can also this sort of geometric distribution is also used in many kinds of drug survey when we are number before bringing a drug to the market, if we are interested in finding out suppose if a drug we should get for a drug to be to bring into the market, it should at least prove that it is 80% of the cases it is successful.

So, we will do go on doing the trial till we find that it is showing 80% successful now, this is people forgetting the 80% successful how many patients we have used this drug. So, if this number of persons is quite large to get 80% successful than this drug, there is some wrong with this drug, then it should not be brought to market we would need some more work on it. So, this geometric distribution has also it is used in that type of situation as well.

Now, coming back to this question, I just remember that example I thought of mentioning it now coming back to this example. So, here the defective rate is given for a certain manufacturing process it is known that on average 1 in every 100 item is defective. So, how much is defective it is given what is the probability that the 5th item inspected is the first defective item found, we 5th item inspected first item we have inspected, it is not defective second we found it is not defective 5th item we found it is defective, what is that probability?

Probability is geometric distribution is simple formula p x q. There is a simple formula p into q, q^{x-1}, where x can take any value x can take any value 1, 2, 3 any value. So, we can find the first effective in the first trial itself. So, this is so here x = 5, p = 0.01, 1 in 10, 100. So, putting in the formula that is what that is the probability of getting 50 item inspected is the first defective found.

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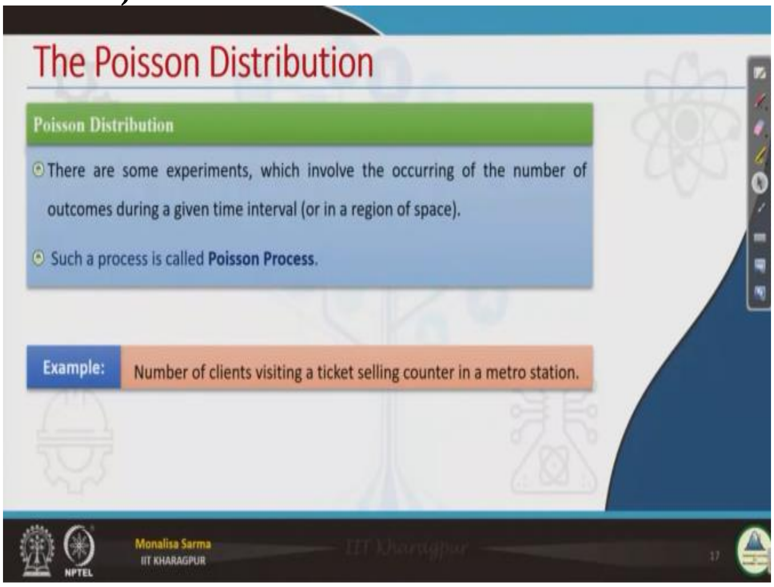

Now, there is one more this is the last probability distribution that we will be discussing under the umbrella of discrete quality distribution. So, Poisson distribution there are some experiments which involve the occurring of the number of outcomes during a given time interval here as in what to say, in contrary to binomial distribution here what we are number of outcomes in a given time interval, there we have trying in binomial distribution or its related negative geometry we are trying to find out the number of successes in so many trials here what is that which is occurring, the number of outcomes during a given interval. A very good example is toll where does what to say the toll collection way for highway we have this toll plaza is where we have to when the car passes, we have to give the toll value so we can say in a span of say 1 hour how many car has passed the toll or in a particular highway in a span of some few hours how many vehicle has passed.

So, or in a ticket counter in a particular time, how many people are there in the counter in a particular time. So, this is similarly suppose in a particular material and material a particular size 1 metre in a material of size 1 metre, how many flaws are there maybe flaws maybe in terms of design flaws, maybe in terms of density of the material whatever in a flaw material of 1 metre, how many flaws are there.

So, that is what which involves the occurring of the number of outcomes during a given time interval or in a region of space. So, such experiments are called Poisson experiment. So, example number of clients visiting ticket selling counter in a metro station as I already mentioned, like here one more thing like number of calls coming to a call centre at a particular time that is also a Poisson distribution.

But again there is something like if you talk about going to ticket counter ticket counter the people arriving and people in the queue for ticket counter and during early morning or during peak time it varies is not it? So, there are again different perception of Poisson distribution to model this where it is dependent on time with a number of people with the average number of people coming to a counter it is dependent on time.

So, the average number of events occurring in a particular time interval. So, if it is dependent on time, then it is not as simple Poisson distribution that we call it is as non homogeneous Poisson distribution, which we will be not be discussing this is a bit higher level. So, just for your knowledge, I just thought of mentioning it now, we are just considering the simple Poisson process.

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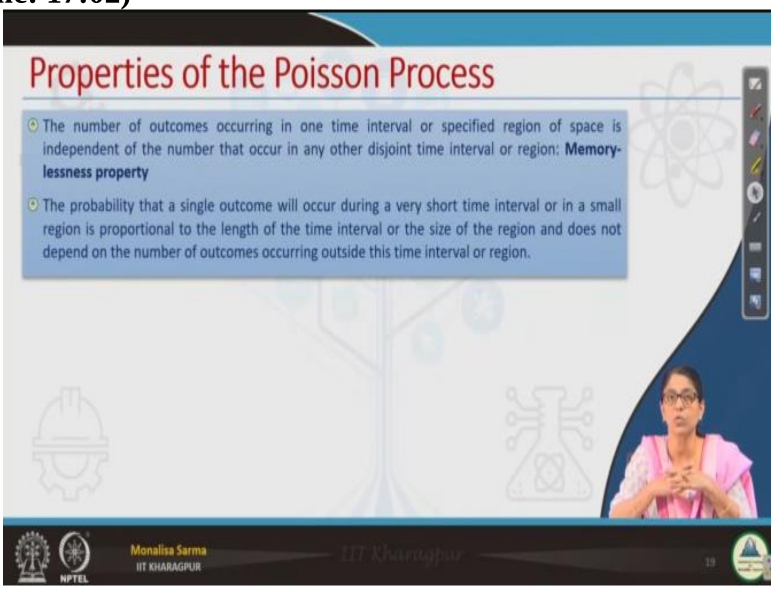

So, Poisson process had certain characteristics certain quality we can say. So, what are those? The number 1 the number of outcomes occurring in 1 time interval or specified region of space is independent of the number that occur in any other disjoint time interval or region what is it? The number of outcomes occurring in 1 time interval or specified region of space is independent of the number that occur in any other disjoint time interval or region.
In a highway if in 1 hour total 100 car has passed that does not mean in the next hour also 100 car will pass or that does not mean already 100 car has passed that means now, only 2 or 3 car will only pass that is not that it is not dependent on whatever what happened in one time interval that will not dictate what will happen in the other time interval. So this property basically this is as if it does not have memory.

So, this property is called memory lessness property. So, again, I am repeating see number of outcomes occurring in one time interval is independent of the number that occurs in the previous time in time, so it does not have any memory. So, next property, the probability that a single outcome will occur during a short time interval or in a small region and in a very short time interval it is proportional to the length of the time interval.

If there is a probability that a single outcome will occur in this highway probability that a single car will pass it is what did that probability actually it is proportional to the length of the time interval there is no other thing that will dictate that it is not that because 1000 cars are passed in the last hour. So, just one 1 will pass it is no not that it is proportional to the length of the time interval or in case of space it is proportional to the size of the region and does not depend on the number of outcomes occurring outside this time interval or range or region.
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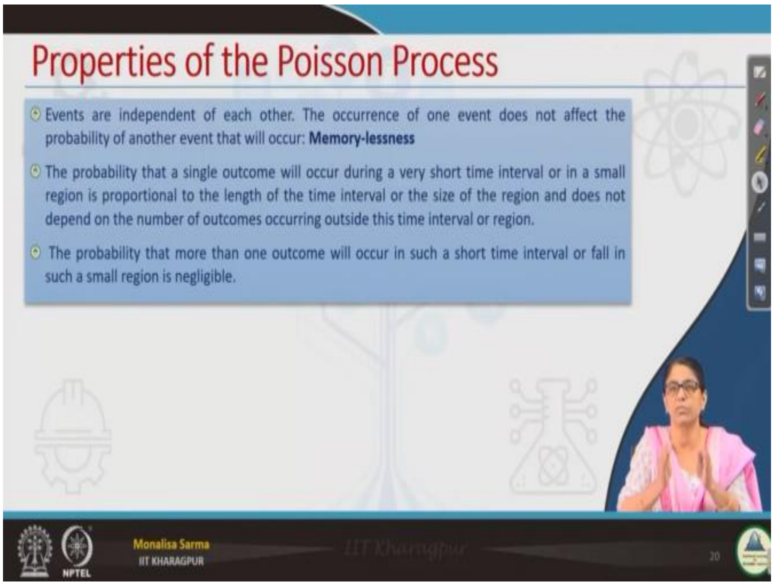

The third property the probability that more than one outcome will occur in a very short time interval or fall in such a small region is negligible. If we consider a very small time interval the more than one outcome will occur is very negligible. So, these are the 3 properties of Poisson distribution.

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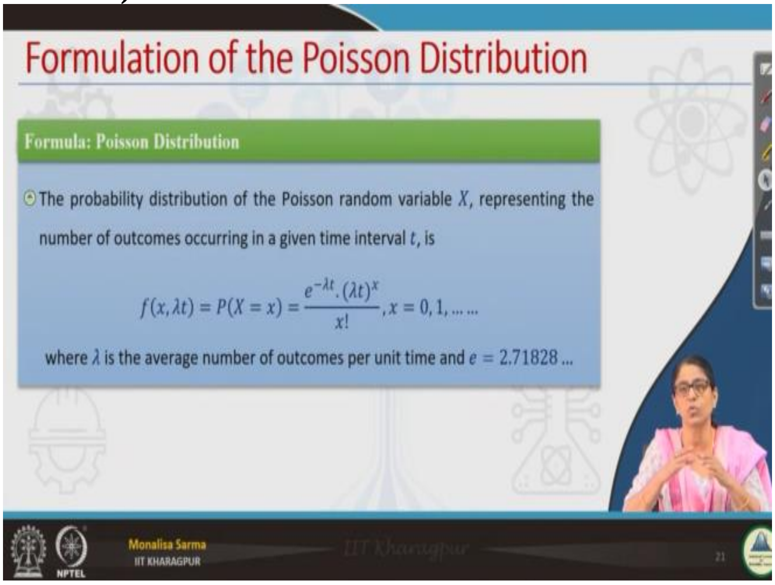

Now, what is the function of Poisson distribution and Poisson distribution we have a parameter that is we call it lambda where lambda is the average number of outcomes per unit time, average number of cars passing through that highway in 1 hour. average number of cars passing the toll station in 1 hour or in 24 hours, whatever it is, so that is the parameter this is lambda.

So x is the number of cars that pass the toll station or number of cars that pass the highway or number of flaws in the material of 1 metre, whatever it is, that is the X that is the random variable, X is a random variable, λ is the average number of outcomes per unit time and t is the total time interval that we are considering. So, λ is given suppose in terms of 1 kilometre, and suppose we need to find out in terms of 10 kilometre.

So, that is t so, what is the formula for that $e^{-\lambda t} \lambda t^{x} / x!$ this also like how we have calculated for uniform probability distribution binomial whatever what we have seen we could usually can easily calculate it from our knowledge of probability theory is not it? What we have learned from probability theory binary permutation combination for finding out the sample space we could easily find out the function.

Similarly, for Poisson distribution also if we go step by step you can easily find out deduce this expression. But that is not necessarily if you are interested you can do it by yourself there is no issue, but this is the function for Poisson distribution f (x λt) is nothing but $e^{-\lambda t} \lambda t^{x} / x!$ these things are so, thing it is so, common it will be the tip of your tongue always once you learn it, you will never forget it.

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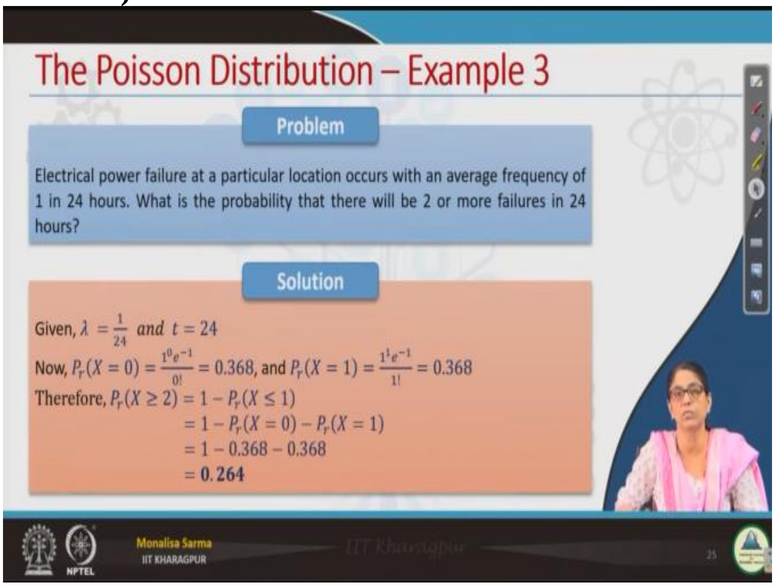

So, now, 1 small example, electrical power failure at a particular location occurs with an average frequency of 1 in 24 hours, what is the probability that there will be 2 or more failures in 24 hours? What it is given every frequency is 1 in 24 hours that means λ is 1 per 24 hours. Now, what we need to find out what is the probability that there will be 2 or more failure in 24 hours? My t is 24 hours and I have found in λ also 1 in 24 hours.

So my λt is just 1 λt is 1. Now my X is a random variable, what is the X? X is the number of failures, I am interested in finding out 2 or more failures that means, what is the probability of finding 2 or more failures? And it is basically same as finding out 1 minus of 0 failure + 1 failure, 0 failure, 1 failure means at most 1 failures, that is 0 + 1, if I subtract 1 minus of that, then I will get 2 or more failures.

Because or else I have to find out 2 failures, 3 failures, 4 failures, it might go to infinity, so that is difficult to find out instead of that I will take the easy path that is 1 - 0 + 1. So what is probability of X = 0? Putting in the formula $e^{-\lambda t} \lambda t^{x} / x!$. So, we found out X = 0, X = 1 we found that X = 1 so for X greater than 2 is 1 minus of these 2 values that is all.

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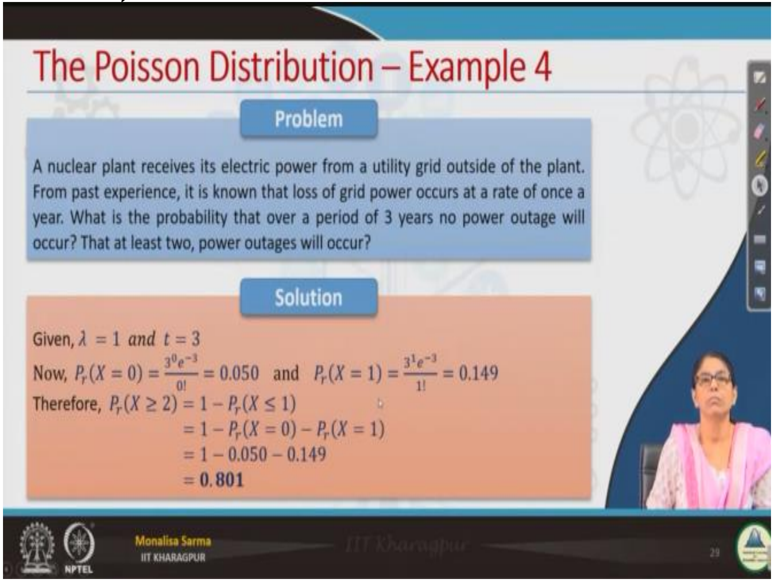

So, 1 more example here we have a nuclear plant receives this electric power from a utility grid outside of the plant. From past experience, it is known that loss of grid powers occurs at a rate of once a year. A nuclear plant receives its electric power from a utility grid outside of the plant, from past experience, it is known that loss of grid occurs at a rate of once a year. So that is λ average is once a year λis 1 per year.

What is the probability that over a period of 3 years no power outage will occur? Now my t is 3 years, λ is 1 per year my t is 3 years. I am interested in finding out what is the probability of a period this is no power test. That means I am interested in finding out probability of X = 0, given λ is 1 and t is 3 simple just put it in a formula λ is 1, t = 3, X = 0.

Now next question is at least 2 power outage will occur at least 2 that means it can 2, 3, 4 anything. So similarly now 1 - 0 + 1 that will give me at least 2 is not it? So that is why I am here see, we found out X = 0 we found out what is X = 1 so 1 - X = 0 - X = 1 will give me X >= 2.

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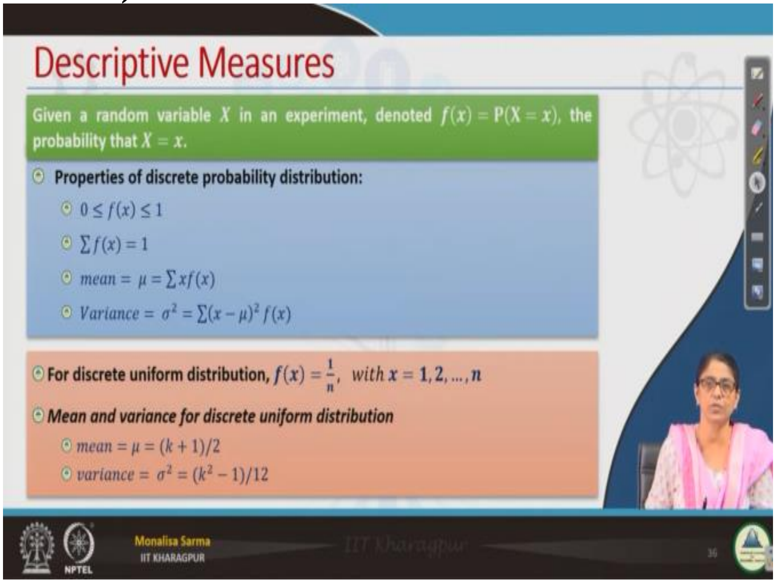

So, now to summarise the descriptive measures, but of all the probability distribution that we have seen, first properties of the discrete probability distribution what we have seen a value of fx lies from 0 to 1 that we have seen what is f(x)? f(x) is the probability at point x at value probability of X. Then summation of f(x) = 1 mean = x f(x) ∑ f(x) variances = ∑ $(x – \mu)^2$ f(x) we have already seen all this now for discrete uniform distribution we have seen f(x) = 1 / n.

If the value that x can take is 1 / n, then if we find out μ= k + 1 / 2 variance is k^2 - 1 / 12 this putting into formula you will get it and again and again you do not have to put it in a formula and evaluate it you can just remember it that is this like a + b^2 we need to do it and solve it and do it we can remember a + b^2 = a^2 + 2 a b + b^2. Similarly, you can dissolve these things also it will be in your memory always.

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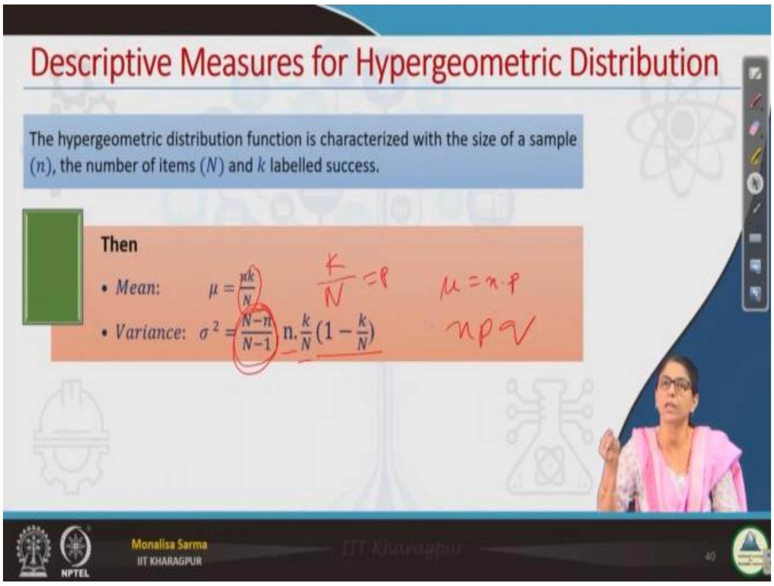

Similarly, for binomial distribution, what is the mean? Mean is np σ variance is np q 1 - p is q. So, for hypergeometric distribution mean is I see here means hypergeometric we have also done 1 problem also mean is what? n x k / N what this k / N symbolises? k / N I am writing it here what is k / N? k / N is nothing but it will be out of n out of capital N k are success. So, what is this? This is p basically is not it?

So, n into so, μ is equal to basically n x p so, hyper geometric distribution mu and binomial distribution μ is same binomial distribution also μ is np here also μ is np because k / N is nothing but p. And in σ also same for binomial distribution what was my σ^2? σ^2 is npq here also you see this is n this is p this is q 1 - k / N is q, but we have just 1 extra term here that is n - n small n / N - 1.

Now, one thing now, if our size of N is very big, size of N is very big capital N and small n is very small, what happen like from a from a what to say, bucket of water if I bring out 1 glass of water does will it make any difference? It will not make any difference. It will be hardly it will make any difference from a bucket of water I am just bringing out one glass of water. But if in a small bowel where I am putting say 2 glasses of water or 1 glasses of water from there, if I pick up 1 glasses of water there, it will make difference.

So, that is what here you can see in the hypergeometric distribution, this value N - n this value, n – N / N - 1 this will make a difference when my capital N is not very large and small n is also not very small, then only it will make a difference else it might capital N is very large and small n is very small, that means N - n / N - 1 it will be almost equivalent to 1 only it will not make much of a difference.

Then it turns out to be same as binomial distribution, why I mentioned this for very large N, large capital N total size of the sample and very small n. So, instead of using hypergeometric distribution we can use binomial distribution as well because we have seen it hardly makes any difference μ is seen σ we have just another 1 extra factor here that is N - n / N - 1, but when N is large or small n is small it differences not reflected at all.

So, we can for very large N and very small value of the sample size that is a small n we can use instead of hypergeometric we can use binomial distribution as well, when the case is suitable for hypergeometric that means without replacement, but still we can use binomial that is an example from a bigger bucket of water bringing one glass will not make any difference.

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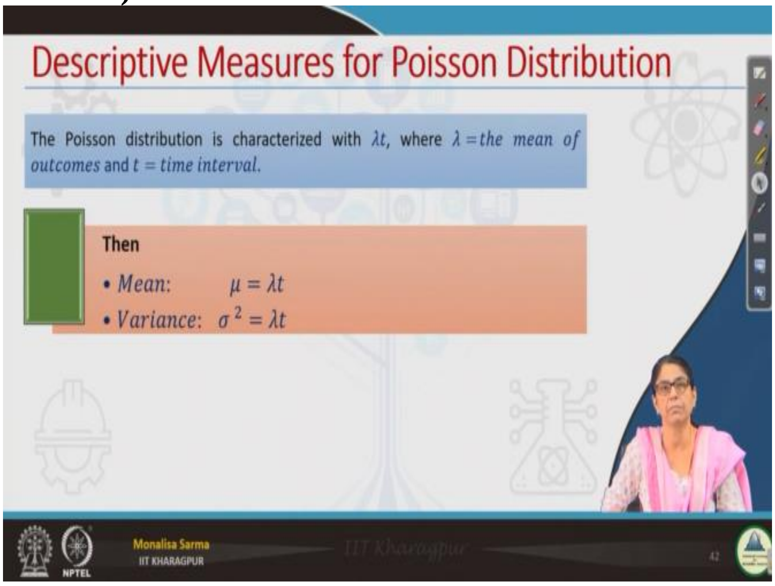

So, similarly for Poisson distribution mean is λt and variance for positive is the mean and variance both are same that is λt.

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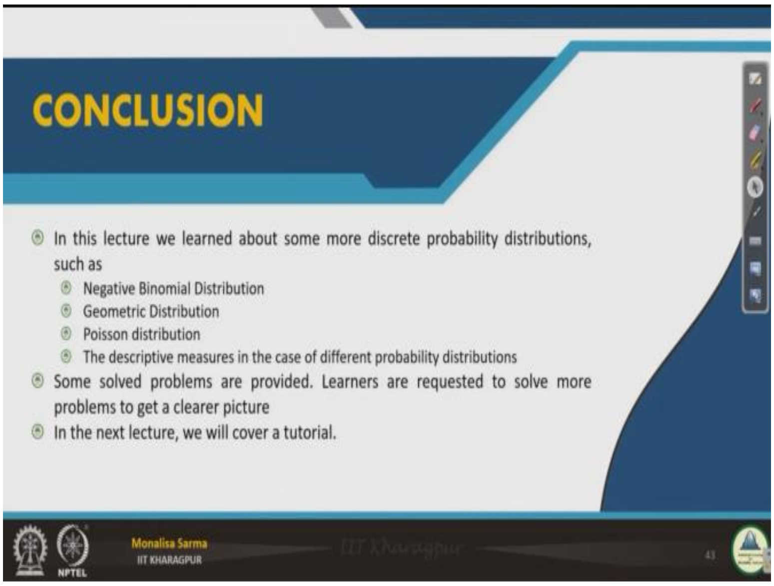

So, now to conclude this lecture in this lecture, we have discussed negative binomial distribution, geometric distribution, Poisson distribution we have also seen the descriptive measure of different probability distribution. Again I request you to solve as many problems as possible, so, that you become confident in all these distributions. And in this next lecture, we will cover a tutorial on this discrete probability distribution and the references and thank you guys.

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