Algorithmic Game Theory
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Lecture - 09
Implications of Minmax Theorem

Welcome. So, we started proving Minmax Theorem. We last time we wrote the linear programs for row player and column player. And we also observed that they are duals of each other and we will see the proof of minmax theorem, using strong duality theorem.

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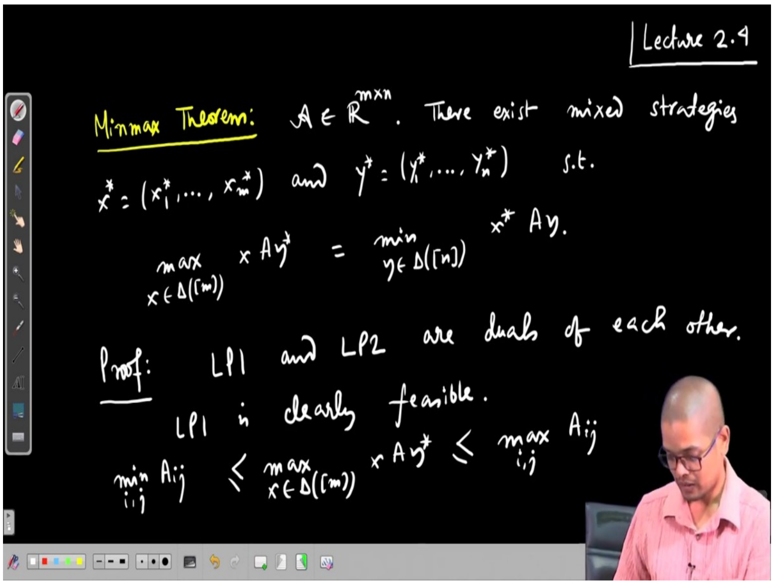

So, let us again recall what is what was minmax theorem. So, it says that, let A be a matrix real matrix$m \times n$ and then, it says that, there exist mixed strategies x^* which is equal to $x_1^*, \ldots, x_m^*$ and y^* which is equal to $y_1^*, \ldots, y_m^*$, such that, $max_{x \in \Delta([m])} xAy^*$ is same as $min_{y \in \Delta([n])} x^* Ay$.

And the proof let us do its proof, we wrote the linear programs for row and column players they were LP1 and LP2 and we first observed that LP1 and LP2 are duals of each other and we want to apply the strong duality theorem, for that we need to the assumption is that, at least one linear program say LP1 must be feasible and bounded.

So, is LP1 feasible? So, LP1 is clearly feasible, because for feasibility x just needs to be a probability distribution take any probability distribution x and that satisfies the feasibility conditions of LP1. Is it bounded? Yes. So, this value $max_{x \in \Delta([m])} xAy^*$, this is less than equal to $max_{i,j} A_{ij}$ and greater than equal to $min_{i,j} A_{ij}$. So, its bounded by maximum and minimum value of the matrix. So, it is so, it satisfies the assumptions of strong duality theorem.

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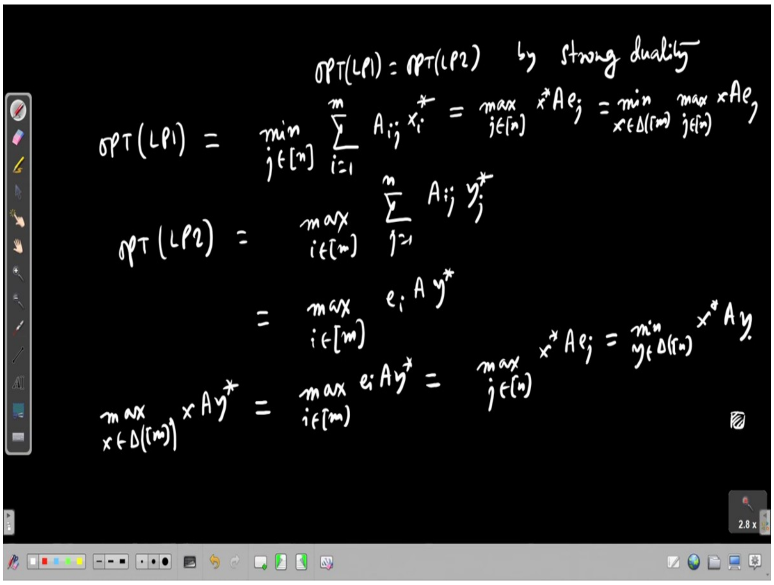

And what we have hence is OPT of LP1 equal to OPT of LP2. Now what is OPT of LP1? Let us write it this is by strong duality. So, what is OPT of LP1? OPT of LP1 is $min_{j \in [n]} \sum_{i=1}^{m} A_{ij} x_i^*$. Suppose, say x equal to x^* is giving is giving the optimal value. So, or maximize maximizes the sum this is $i=1,\ldots,m$. And suppose, $y=y^*$ optimizes LP2.

So, OPT LP2 is equal to $max_{i \in [m]} \sum_{j=1}^{n} A_{ij} y_j^*$ ok, and this you can also write as this is equal to in matrix multiplication notation max $i \in [m]$, e i e i is the row vector with ith coordinate 1 and everything else 0 $e_i A y^*$, and the upper one you can write it as $max_{j \in [n]} x^* A e_j$.

Now, because x^* minimizes this quantity. So, we can write it as this same as $min_{x \in \Delta([m])} max_{j \in [n]} x A e_j$. Similarly, because y^* minimizes that quantity this is. So, let us see. So, what does the minmax theorem needs to prove? So, minmax theorem, you need to show that $max_{x \in \Delta([m])} x A y^*$.

Now, this is same as max yes, $max_{i \in [m]} e_i A y^*$, which is same as OPT of LP2 and by strong duality OPT of LP2 equal to OPT of LP1. So, this is $max_{j \in [n]} x^* A e_j$ which is nothing but, this is $min_{y \in \Delta([n])} x^* A y$. So, it says that you fix y^* and try and see by varying x star by varying x, what is the maximum value of $x A y^*$? Is same as you fix x star and by varying y you take the minimum value and these two coincides.

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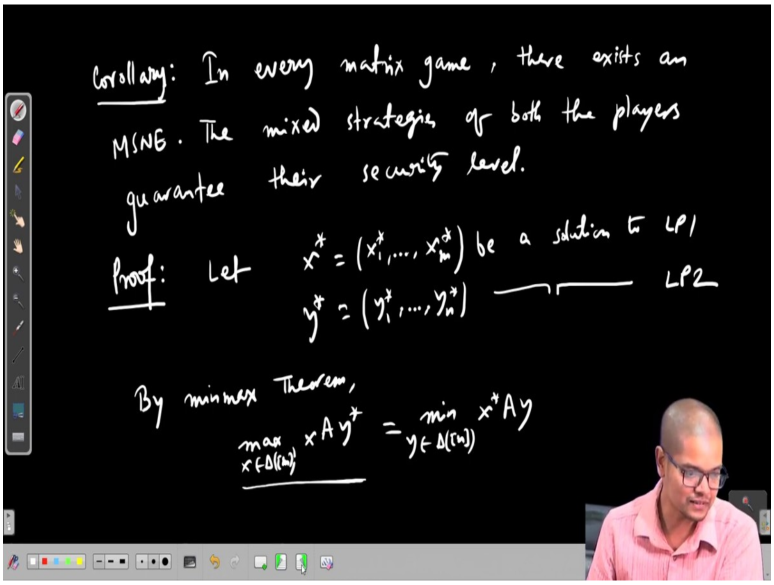

So, now we will have beautiful corollaries from here. The first corollary is that, in every matrix game matrix game, there exists an MSNE. This follows immediately from Nash theorem, but we have not proved Nash theorem, and independent of the without using Nash theorem, this follows from minmax theorem. We will see and not only that, the mixed strategies of both the players guarantee their security level ok.

So, what is it? So, let us prove it. So, let$x^* = (x_1^*, \ldots, x_m^*)$, be a solution to LP1. Solution means, it provides optimal value for LP1 that is one x^* and y^* is an optimizer for LP2 ok. And by minmax theorem, we have seen that, this is, by minmax theorem, if I look at $x A y^*$ and try to max let column player play y^* and let x^* let the row player maximize over $x \in \Delta([m])$ and let row player play x^* and let column player minimize $x^* A y$, these two values are same. Now, you see that we first claim that (x^*, y^*) is a mixed strategy Nash equilibrium.

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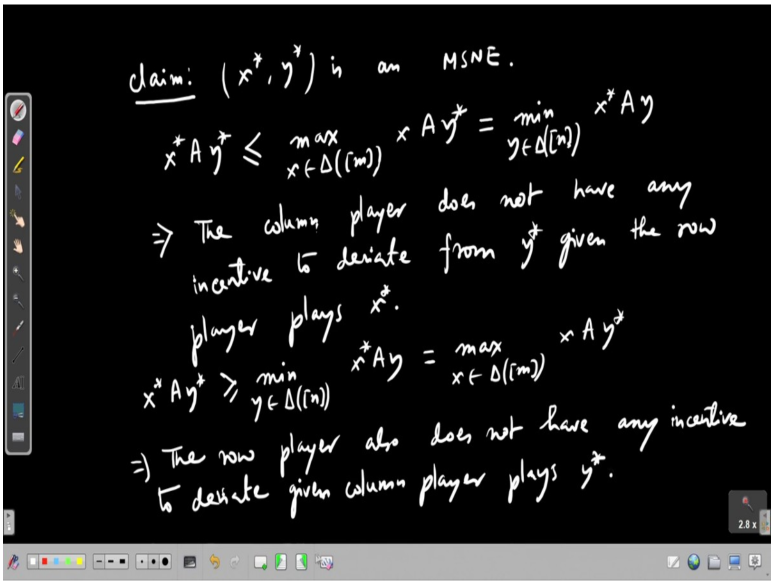

Let us see, first claim (x^*, y^*) is an MSNE ok. Let us see why, what is $x^* A\, y^*$? If you look at this x^* maximizes this because x^* is a solution to LP1 in particular x^* maximizes this quantity. So, x^* is, $x^* A\, y^*$. So, from row players perspective, because x star is a solution to LP1, $x^* A\, y^*$ is less than equal to $max_{x \in \Delta([m])} x\, A\, y^*$ ok.

But, this is nothing but by applying minimax principle, minmax theorem, this is $min_{y \in \Delta([n])} x^* A\, y$. So, this shows, hence the column player does not have any incentive to deviate from y^* given the row player plays x^*. Why it is so? Because, the column players utility matrix is -A and when $x^* A\, y$ is minimized, then the utility of column player is maximized.

Similarly, you can also look at $x^* A\, y^*$, y^* is again a solution to LP2, this is greater than equal to $min_{y \in \Delta([n])} x^* A\, y$ and which is equal to by minmax theorem, $max_{x \in \Delta([m])} x\, A\, y^*$. So, similarly, the row player also does not have any incentive to deviate given column player given the column player plays y^*.

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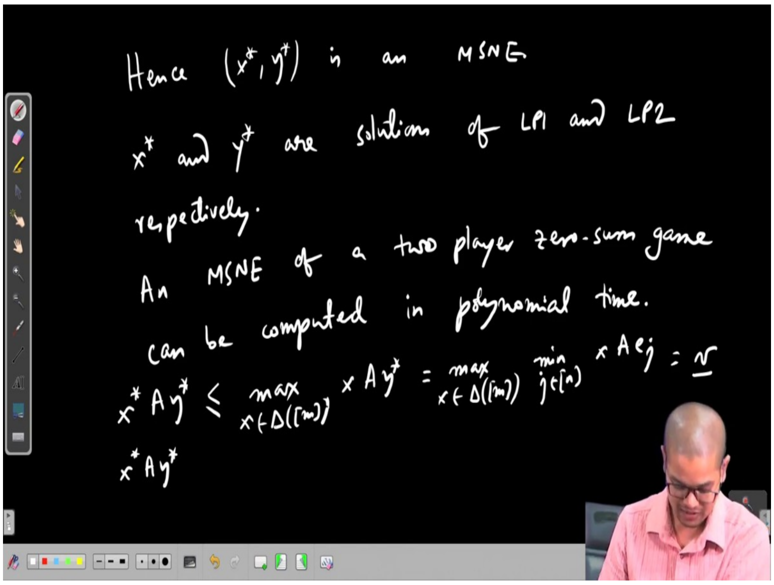

Hence, (x^*, y^*) is an MSNE ok. And why does, and also you observe that, you know x^* and y^* are solutions to linear programs; x^* and y^* are solutions of LP1 and LP2 respectively. And because, linear program can be solved in polynomial time, they can be solved efficiently an MSNE of a two player zero-sum game can be computed in polynomial time ok.

And of course, we there every player is guaranteed their security value and from where we get this? We get this from this equality. So, what is the utility of column row player? It is $x^* A\, y^*$, and we have seen that this is greater than equal to $max_{x \in \Delta([m])} x A\, y^*$ and what is y^*? So, replace the definition of y^* x in y^* is a solution for linear program two.

So, that means, this is $min_{j \in [n]} x A e_j$ this is nothing but, the value of row player; among all mixed strategies of row player, you pick the one you iterate over all mixed strategies of row player and see what is the minimum utility that is guaranteed, in this particular mixed strategy of row player.

Similarly, for column player, you see that $x^* A\, y^*$ is less than equal to sorry, what do we have? This should be greater than? Yes, this should be greater than, then only it make sense. This is greater than and now similarly for column player $x^* A\, y^*$, no, this is just a minute, this is less than, yes. So, $x^* A\, y^*$ the utility of row player is at most.

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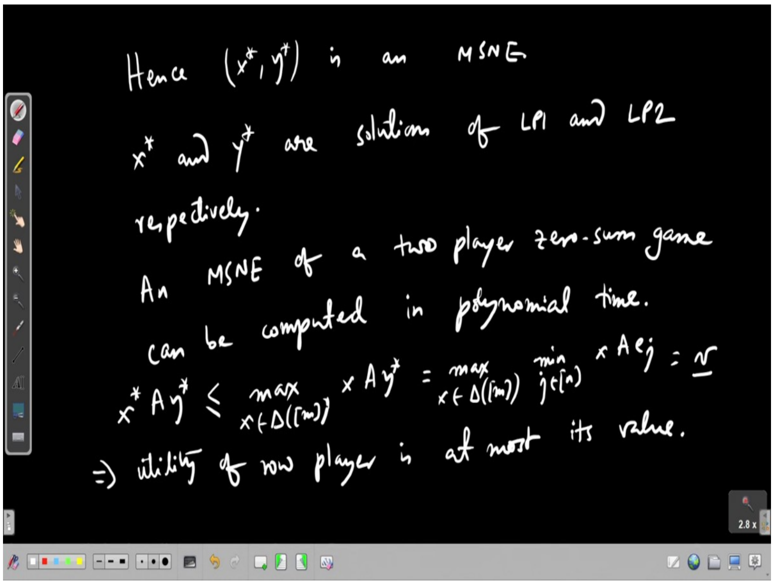

So, from here we conclude that, utility of row player is at most its value. But we have already we already know that the utility of row player in any mixed strategy Nash equilibrium must be at least its value.

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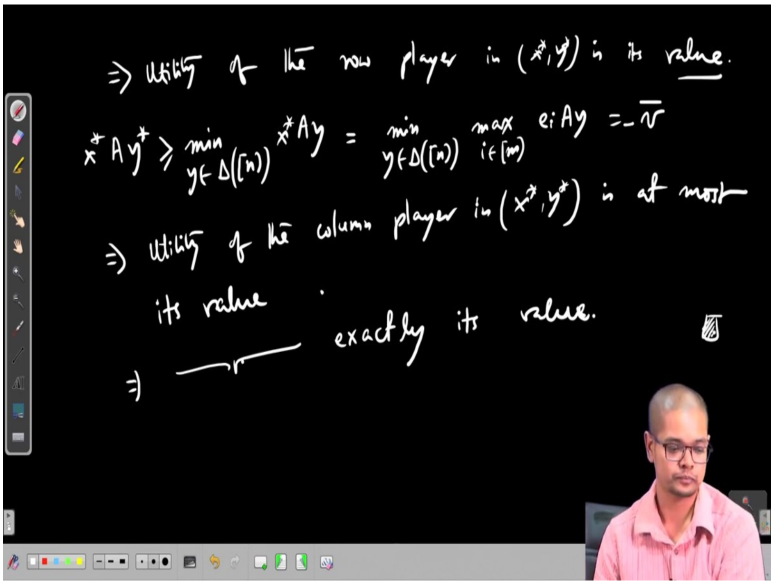

So, in particular, we can conclude that, the utility of the row player in (x^*, y^*) is exactly its value is its value. Similarly, for column player. So, for column player we should use the other inequality $x^* A\, y^*$, this is greater than equal to $min_{y \in \Delta([n])} x^* A\, y$ which now what is x^*? x^* is the minimum $y \in \Delta([n])$.

Now, x^* was an optimal solution of LP1. So, this is $max_{i \in [m]} e_i A y$. So, this is nothing but the value of the column player and this is minus. So, because the utility matrix of column player is -A, the utility of the column player in (x^*, y^*) is at most its value and because utility of any player in an MSNE should be at least its value, this is exactly its value.

This, what we have claimed before, that in general game in non-zero sum game, it may be the case that the utility of a player in a mixed strategy Nash equilibrium, is strictly more than its value. But in two player zero sum game, you take any mixed strategy Nash equilibrium the value the utility of that of both the players in this mixed strategy Nash equilibrium is exactly their value, ok.