Algorithmic Game Theory
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Lecture - 08
Minmax Theorem

Ok. Welcome. So, in the last class we have defined security level and at the end we showed that in any normal form game in any mixed strategy Nash equilibrium all the players are guaranteed their security level.

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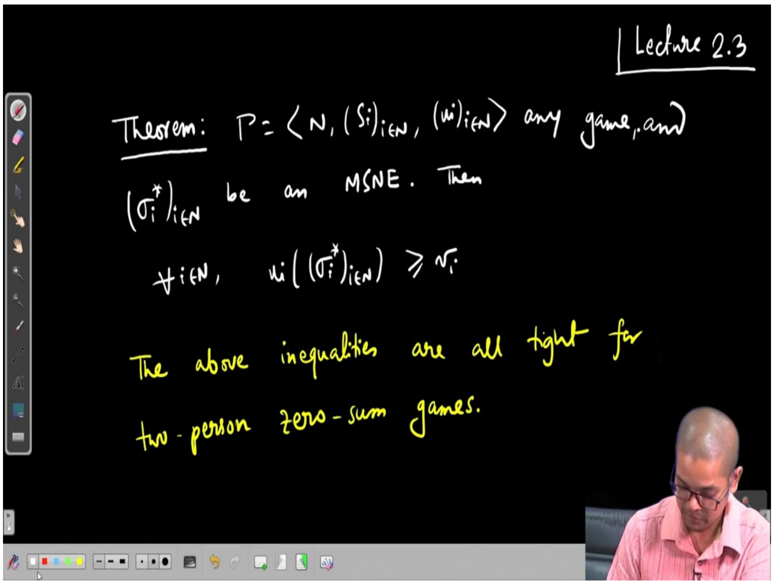

So, this is the theorem that we proved in the last class. Let Γ equal to $\langle N,(S_i)_{i\in N},(u_i)_{i\in N}\rangle$ be any game in normal form, then not then and $(\sigma_i^*)_{i\in N}$ be an MSNE: Mixed Strategy Nash Equilibrium then for all player $i\in N$, the utility of player i in this mixed strategy profile is greater than equal to v_i.

And, the proof is obvious its it directly follows from the definition, but the beautiful thing about zero sum game, two person zero sum game is that the above inequalities are all tight for two person zero sum game. So, for two person zero sum games, the utility that each player gets is exactly their value not more or not less. So, that is the main theorem that which we will prove for two person zero sum game.

So, let us see; so, towards that let me write what are the what are the values of row player and column player in a two person zero sum game.

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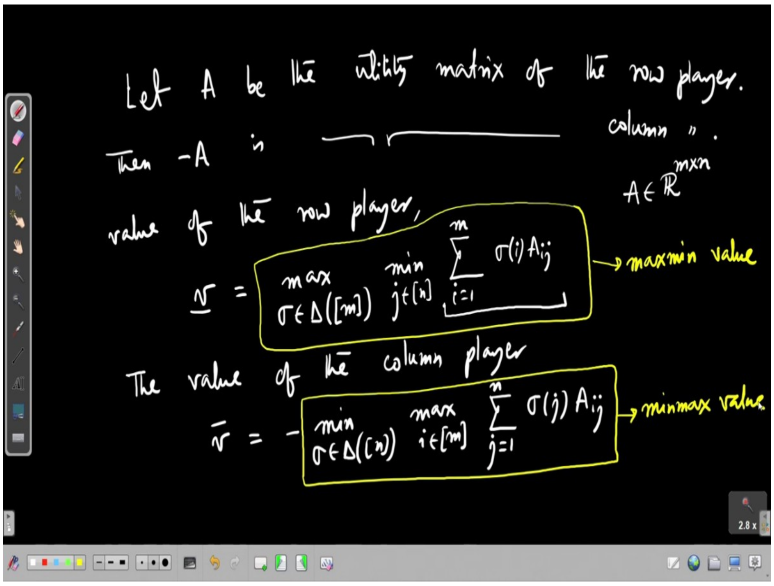

So, let A be the utility matrix of the row player, then by definition minus A is the utility matrix of the column player. So, what is the security value of row player? So, let me write this way value of the row player v underscore is max sigma in delta m. So, assume A is an $m \times n$ matrix. So, row player has m strategies and column player has n strategies. So, $max\, min_{j\in[n]} \sum_{i=1}^{m} \sigma(i) A_{ij}$.

So, what is this? Think of the row player is playing the mixed strategy sigma and it is considering what worst can happen. So, if column player plays j, then $\sum_{i=1}^{m} \sigma(i) A_{ij}$ is the utility that player the row player gets by playing the mixed strategy sigma and column player when column player plays j. Now, among all possible js the row player looks at what is the minimum utility that is guaranteed and it is trying to maximize it. So, this is the value of row player.

Similarly, the value in mixed strategies. So, again if you see that we have we are written sup. Now, this sup can be replaced in by max because this supremum is what is called attend, we will come back to it; for this finite sum this can be written as max. So, similarly the value of the column player, which is $\bar{v}$ is you see column players column players utility is -A.

And so, if you write that then it will become minimum $\sigma \in \Delta([n])$, minimum over the mixed strategies of column player $max_{i \in [m]} \sum_{j=1}^{j} \sigma(j) A_{ij}$. This is the value of the column player minus of this; this is the value of the column player. The first term is called the max min value and the second term without negative is called the min max value.

So, this is called max min value of a matrix game and this is the min max value ok. So, we will show that the max min value coincides with the min max value, so, max min value equal to min max value.

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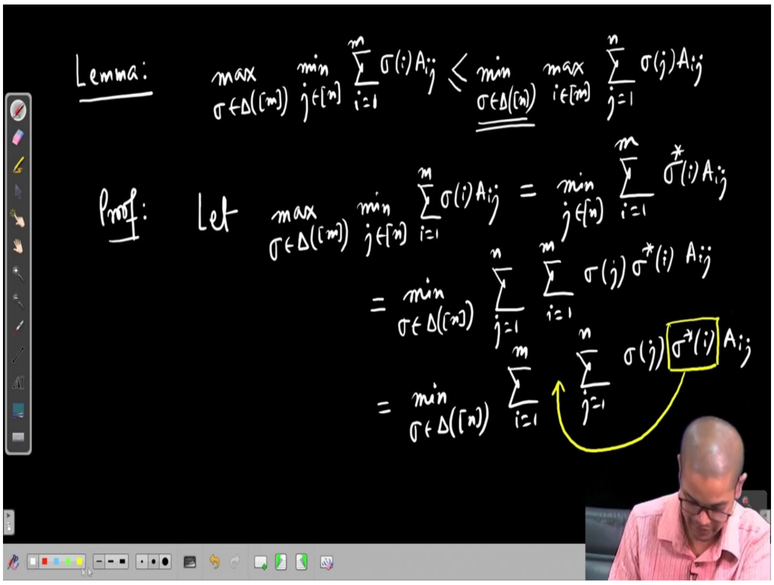

But, one direction is very easy; lemma that max min value max $max_{\sigma \in \Delta([m])} min_{j \in [n]} \sum_{i=1}^{m} \sigma(i) A_{ij}$. This is the max min value max min value is always less than equal to min max value $\sigma \in \Delta([n])$ very easy. So, let see the proof. The other direction will need more work and when we show the other direction, we get max min value equal to min max value ok.

So, let us start with the left-hand side and let us consider we what value of sigma, say call it σ^* maximizes the quantity. So, let this quantity is maximized $\sigma \in \Delta([n])$, suppose σ^* maximizes this. So, this is equal to then $min_{j \in [n]} \sum_{i=1}^{m} \sigma^*(i) A_{ij}$ok. So, continuing this so, what is with this minimum again because of convexity we need to go here, we need to replace minimum over j in n with minimum over $\sigma \in \Delta([n])$.

So, we use convexity to write the right-hand side to be equal to $min_{\sigma \in \Delta([n])} \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma(j) \sigma^{*}(i) A_{ij}$. Now, what I do is that I exchange these two double sum. Whenever we have a double sum, it is often convenient or it is often useful to exchange the double sum and see what we get. Now, you see that this particular term $\sigma^{*}(i)$ is independent of j. So, what we do is that we take this sum, take this term and put it outside the inner sum.

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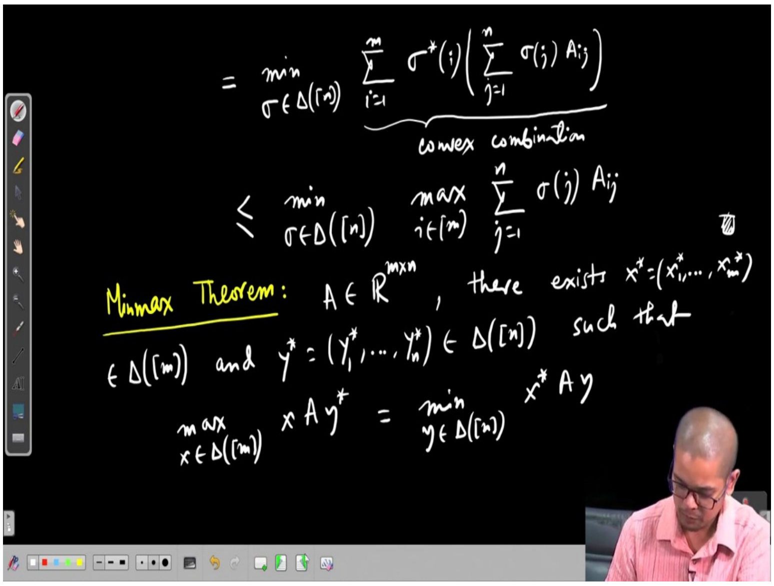

So, what we get is $min_{\sigma \in \Delta([n])} \sum_{i=1}^{m} \sigma^{*}(i) \sum_{j=1}^{n} \sigma(j) A_{ij}$ ok. Now, again you see that this entire sum is a convex combination of inner terms so, this inner this sum will be maximized if I take the maximum term. So, this is minimum, this is less than equal to $min_{\sigma \in \Delta([n])} max_{i \in [m]} \sigma^{*}(i) \sum_{j=1}^{n} \sigma(j) A_{ij}$. This is exactly what we need to prove.

So, max min value is always less than equal to min max value. Now, we will prove the other direction. And so, what is the theorem? Let us write this is called min max theorem. What is it? It says that for every matrix $A \in R^{m \times n}$ there exist two mixed strategies x^{*} which is equal to $x_{1}^{*}, .., x_{m}^{*}$. This is in, these are probability distribution over rows x^{*} and y^{*} equal to $y_{1}^{*}, .., y_{m}^{*}$.

This is a probability distribution over columns such that $max_{x \in \Delta([m])} x A y^*$ is equal to $min_{y \in \Delta([m])} x^* A y$, this is a matrix product. Now, you see for this product to make sense, you know y^* must be a column vector and x must be a x^* must be a row vector, same with x and same with y. So, let us make those assumptions that x^* are row vectors and y^* are column vectors and let us not keep writing transpose and those sort of things ok.

So, we need to we need linear programming duality to prove this theorem. So, towards that let us prepare.

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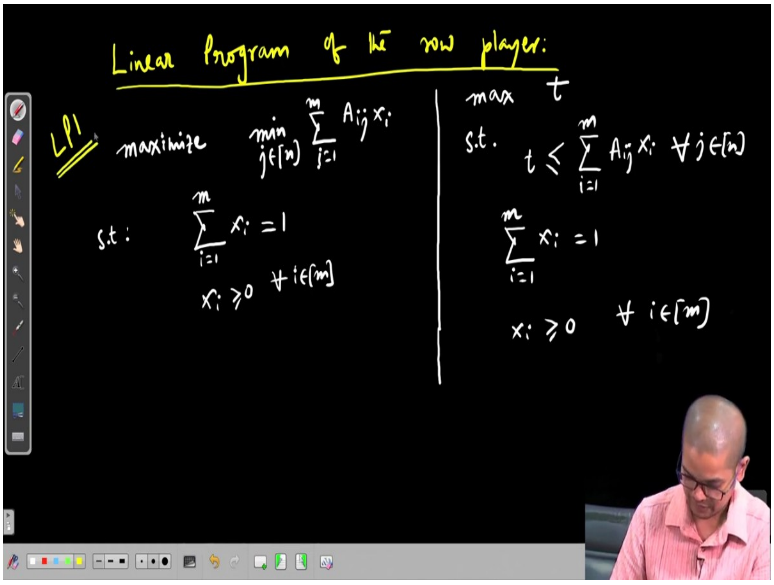

So, let us write the linear program of the row player. What does the row player want to do? Linear program of the row player: What is row players linear program? Row players want to maximize linear program to compute the max min value of the row player. Row player wants to wants to find the maximum of $min_{j \in [n]} \sum_{i=1}^{m} A_{ij} x_i$. So, this is the utility that the row player has row player gets when it plays the mixed strategy x whose components are $x_1, x_2, ., x_m$ and column player plays j.

So, this x_i's must be a valid mixed strategy for that we need that summation, this is subject to $\sum_{i=1}^{m} x_i = 1$ and $x_i \geq 0$ for all $i \in [m]$. Now, this is a linear program. Now, what is a linear program? Linear program is a program which looks like maximize or minimize a linear function of the variables subject to some linear constraints. What is a linear constraint? A linear inequality, a linear function of the variables is greater than equal to or less than equal to or equal to some constant that is a linear constraint.

But, it does not look like this apparently because it maximizes not a linear function of the variables, but it maximizes a minimum of some linear functions which is not linear. But, you know this program can be converted to a linear program to an equivalent linear program as follows by introducing a variable called t and the idea is that t should take value, which is minimum of these things.

So, we maximize t subject to $t \le \sum_{i=1}^{m} A_{ij} x_i$ and this should hold for all $j \in [n]$. You see that if t has to be simultaneously less than equal to $\sum_{i=1}^{m} A_{ij} x_i$ for all j and we have we are trying to maximize t. So, the value of value that the t must take is the minimum of minimum of our $j \in [n]$ $\sum_{i=1}^{m} A_{ij} x_i$. So, this is the one condition and the other conditions of course, remain as it is $\sum_{i=1}^{m} x_i = 1$, $i = 1, \ldots, m$ and $x_i \ge 0$ for all $i \in [m]$.

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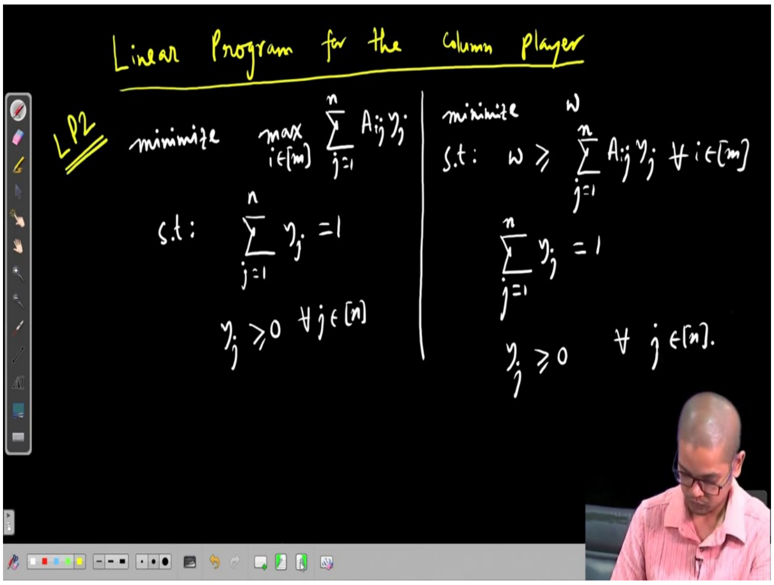

Similarly, we can write down the linear program for the column layer. What is the linear program for the column player? Column player tries to minimize, why minimize? Because, the utility of column player is -A. So, minimize $max_{i \in [m]} \sum_{j=1}^{n} A_{ij} y_j$. So, think of column player is playing the mixed strategy $y_1, \ldots, y_n$ and it is considering each strategy of the row player i equal to $i = 1, \ldots, m$.

And, see what is the what is the worst utility that is guaranteed and that will maximum because the utility of the column player is minus e, this is the subject to this y is must be a probability distribution. So, $\sum_{j=1}^{m} y_j = 1$ must be equal to 1 and $y_j \geq 0$ for all $j \in [n]$, as usual we use that same trick of introducing a variable to write it as a linear program.

So, minimize w subject to$w \geq \sum_{j=1}^{n} A_{ij} y_j$ and this should hold for all $i \in [m]$. And of course, the other conditions remain as it is; that means, summation$\sum_{j=1}^{m} y_j = 1$and $y_j \geq 0$. This should hold for all $j \in [n]$ ok. Now it turns out that; so, let me call these two linear programs, give it some name. This is called say LP 1 and this is LP 2.

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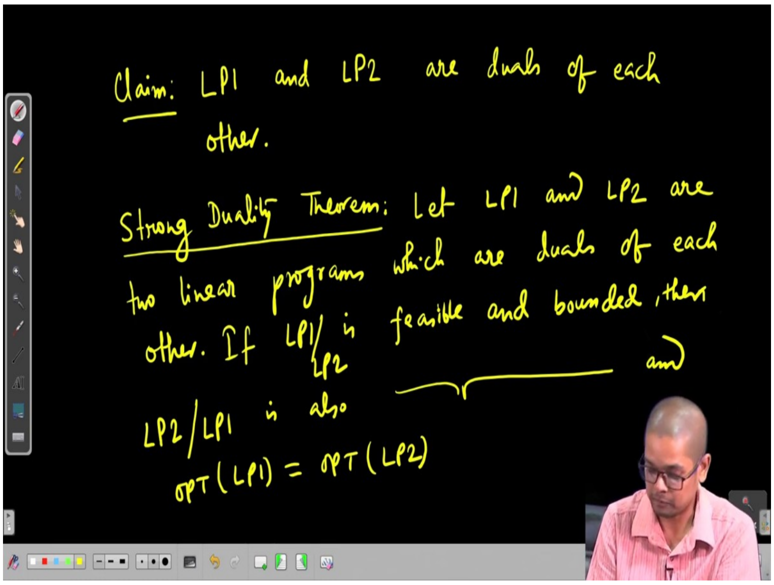

Now, what is the claim? I will leave you to check that LP 1 and LP 2, LP 1 and LP 2 are duals of each other. Now, what is dual? For that you need a linear programming background which I will assume. So, if you do not have some linear programming background, you need to pick that up; we will share the material. So, it is these are duals of each other and you need to know what is linear program and how to solve it and what are the duals.

And, we will also be needing the strong duality theorem which I will state without proof which is again from linear programming. Strong duality theorem that let LP 1 and LP 2 are two linear programs which are duals of each other. Then if LP 1 is feasible; that means, there exist at least one setting of the variables, which satisfy all the constraints and bounded; that means, the maximum or minimum is attained. Then LP 1 or LP 2 either one of them, then the other; then if LP 1 is feasible and bounded then LP 2 is also feasible and bounded.

Or, if LP 2 is feasible and bounded then LP 1 is also feasible and bounded is also feasible and bounded and their optimum values are same OPT of LP 1 equal to OPT of LP 2. So, we will use this strong duality theorem and apply it to the linear programs of the row and column players and we will derive the min max theorem ok.

Thank you.