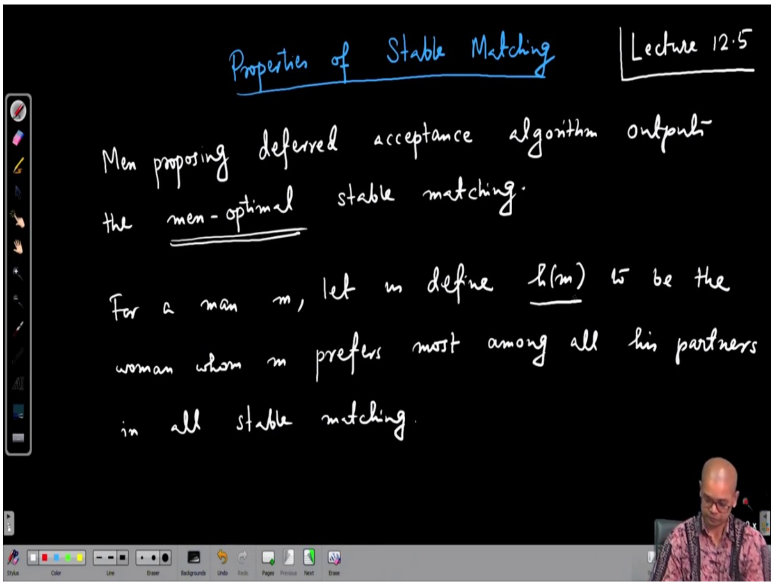
Algorithmic Game Theory
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 60
Properties of Stable Matching

Welcome. In the last lecture we have studied the Gale Shapley algorithm and the men proposing deferred acceptance algorithm and we have formally proved its correctness. So, in today's class we will discuss some more interesting Properties of Stable Matching.

(Refer Slide Time: 00:40)

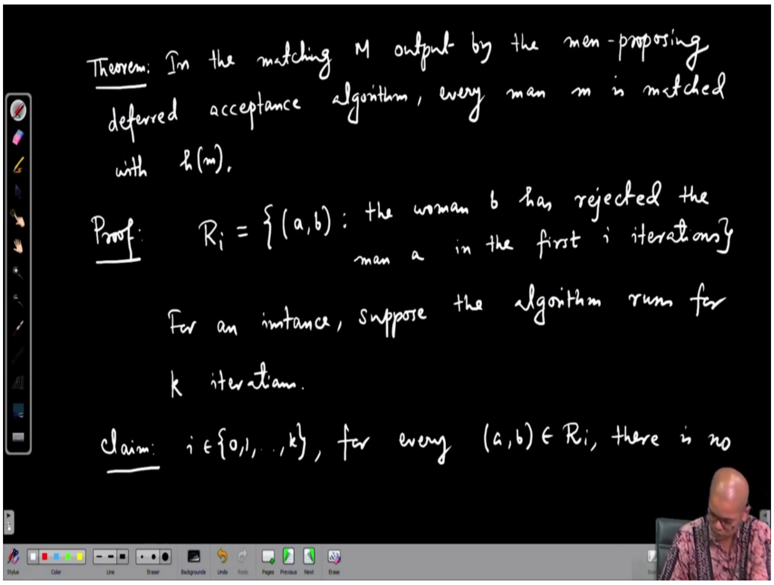


So, properties of stable matching; so, our first big result is that the men proposing deferred acceptance algorithm outputs the male optimal stable matching. Let me right, men proposing differed acceptance algorithm outputs the men optimal stable matching. What do you mean by men optimal stable matching?

Is it that you know all men is matched with his most preferred women? No, that may not be a matching itself. Now what do we mean by men optimal stable matching in what sense it is optimal. So, let us define it. So, for a man m, let us define $h(m)$ to be the woman for a man m let us define $h(m)$ to be the woman whom m prefers most among all his partners in all stable matching's.

In a stable matching instance the stable matching need not be unique in a stable matching instance there can exist many stable matching's and in different stable matching's this man m may get matched with different women. So, among all those women whom he gets matched with in different stable matching's let $h(m)$ be the woman whom he preferred most.

(Refer Slide Time: 04:24)



So, here is a theorem that in the matching M output by the men proposing deferred acceptance algorithm. In the matching M output by the men proposing deferred acceptance algorithm every man m is matched with $h(m)$. The best possible partner that m can get in any stable matching.

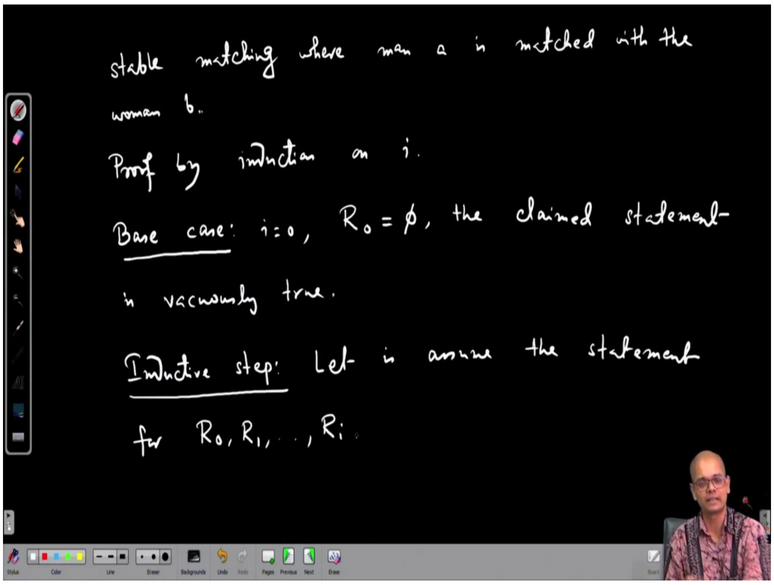
So, first of all you see that it is not even clear that if I match m with $h(m)$ for all men that it is a matching forget stable matching and that is the beauty of this theorem. It says that it is not only a matching this is a stable matching and this is the stable matching output by the main proposing deferred acceptance algorithm.

So, the in that sense this matching is simultaneously best possible stable matching for all the men. It is simultaneously best for all the men. Proof: So, we define it is a it is a proof by induction on the number of iterations. So, define a set R_i to be the set of rejection pairs, say (a, b) such that the woman b has rejected the man a in the first i iterations.

Now you recall that an woman can reject a man in two ways once when a man is so for example, a woman b can reject a man a in two ways when a is proposing b then at that time b may be matched with some other men whom she prefers more than a then b rejects a in that iteration only or b may be matched with a and in the later iteration in some iteration b receives a proposal whom she prefers more than a at that iteration b can reject a.

So, R_i is the set of all rejection pairs that has happened in the first i iterations and R or suppose for an instance for an instance suppose the algorithm runs for k iterations. So, what is the inductive statement that we will prove? So, claim for $i \in \{0, 1, \ldots, k\}$ for every paired $(a, b) \in R_i$.

(Refer Slide Time: 10:09)

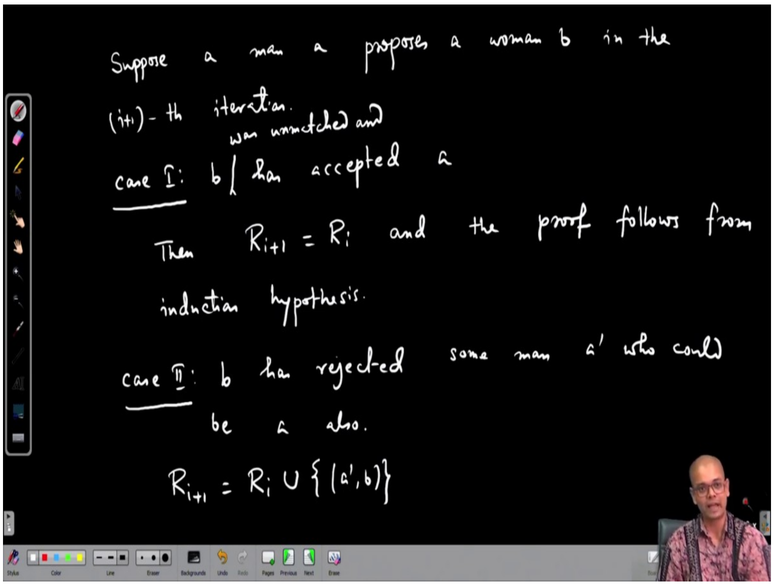


There is no stable matching where man m man a is matched with the woman b. So, this statement we will prove and that proves the, because you know if we leave the rejected pairs then and the this rejected pairs then what. So, for a man a if we leave all her rejections then among those women whom he whom or who has not rejected him he that that man a is getting the best woman. So, that is $h(m)$ or $h(a)$.

So, it is enough to prove this claim. So, first it is enough to prove this claim. So, we will prove this prove by induction on i. Base case for $i=0$ R_0 is empty set and the claim the claimed statement is vacuously true simply because there is no pair a b in the set R_0

because it is an empty set and hence the statement is vacuously true. So, now, the inductive step. So, let us assume the statement for $R_0, R_1, \ldots, R_i$ and we will proof for R_{i+1} . So, let us investigate what has happened in the $(i+1)$-th iteration.

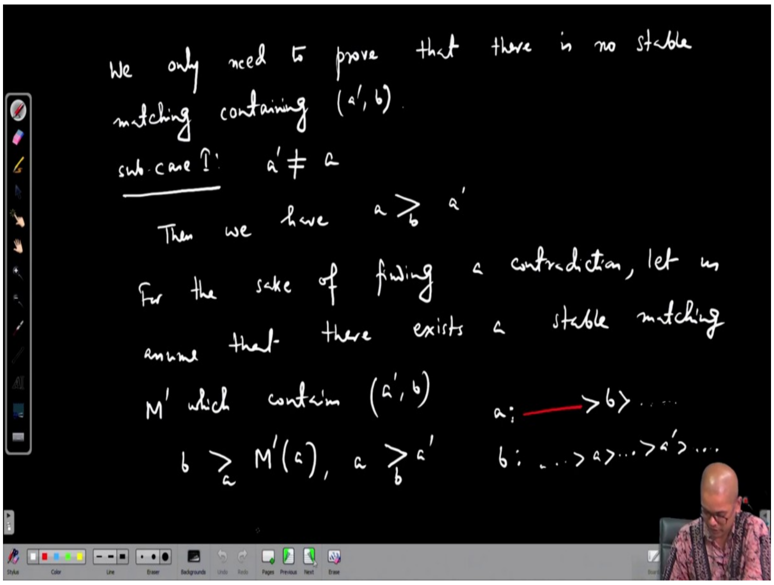
(Refer Slide Time: 13:25)



So, unmatched man must have proposed a woman. So, suppose a man a proposes a woman b in the $(i+1)$-th iteration. There are few cases case 1: b has accepted a easy case b has accepted a then we have $R_{i+1} = R_i$, there is no new rejected pair in the $(i+1)$th iteration. So, there is nothing to proof and because the statement already holds true for R_i and because $R_{i+1} = R_i$ the proof follows from induction hypothesis inductive hypothesis.

And the statement or the proof follows from induction hypothesis. Case 2: b has rejected some man b has accepted a and this is important b was unmatched and has accepted a and the other case was that b was matched. So, b will either reject a or b will reject her existing partner. So, b has rejected some man say a prime which who could be a also. So, if b was matched b will either reject a or her old partner say a prime someone. So, in that case we have $R_{i+1} = R_i \cup \{(a', b)\}$.

(Refer Slide Time: 17:25)



So, we only need to prove that in no stable matching a prime can be matched with b. So, because for all other pairs we have we get what we need to prove from induction hypothesis because they belong to R_i. So, we only need to prove that there is no stable matching containing the pair (a',b) ok. So, again 2 cases, case 1 sub case 1: $a' \neq a$; that means, b has rejected her old partner then we have b prefers a over a' ok and it is a proof by contradiction.

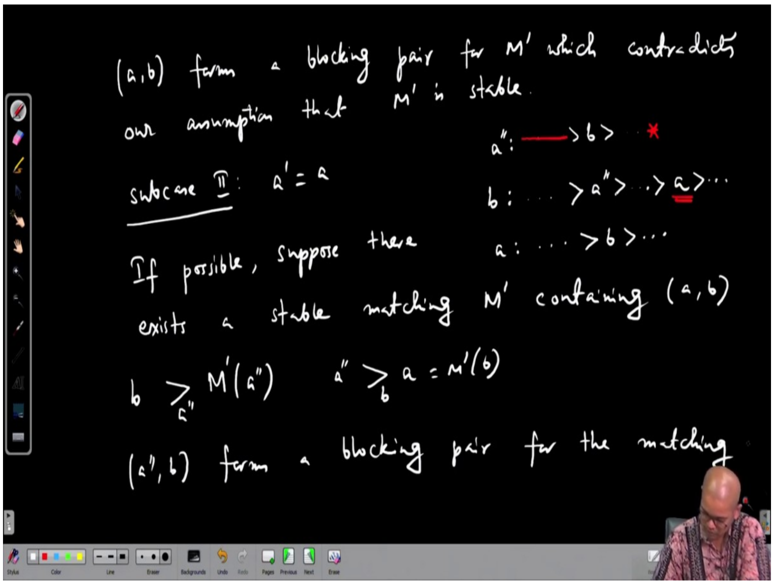
So, for the sake of finding a contradiction let us assume that there exists a stable matching m prime which contains (a',b). Now you see the preference profile of a and preference profile of b. Now in the preference profile of a b is matched somewhere here.

Now from induction hypothesis it follows that a cannot be matched with any woman here because they are all this a has already proposed all the women whom he prefers more than b and got rejected in the first i iterations and hence all those pairs are in R_i and from induction hypothesis we get that a cannot be matched in any stable matching in particular a cannot be matched with in the stable matching m prime with any woman whom a prefers more than b.

But, b is matched with a prime. So, a for b here is somewhere a' and some a is somewhere here. So, b is matched with matched with a'. Now you see that the pair look at ok so if a is not matched with any woman better than b preferred over b and a is not matched with b also in $a^{'}$ because b is matched with $a^{'}$.

The look at as preference a prefer b over his partner in m prime because then if a is not preferred with any woman who he prefers over b and a is not matched with b then a must be matched with some woman whom he prefers less than b. So, and that woman is say $M^{'}(a)$.

(Refer Slide Time: 22:41)



So, a prefers b over $M^{'}(a)$. How about b? b is matched with $a^{'}$ in M', but b prefers a over m prime a over $a^{'}$. So, this shows that (a,b) forms a blocking pair for M', a comma b forms a blocking pair for M' which contradicts our assumption that M' is stable.

We have assumed that M' is a stable matching, but if it is stable matching how can there exist a blocking pair. So, this sub cases is easy the other sub case is also easy the other sub case is a prime equal to a sub case two $a^{'}=a$; that means, b rejects a.

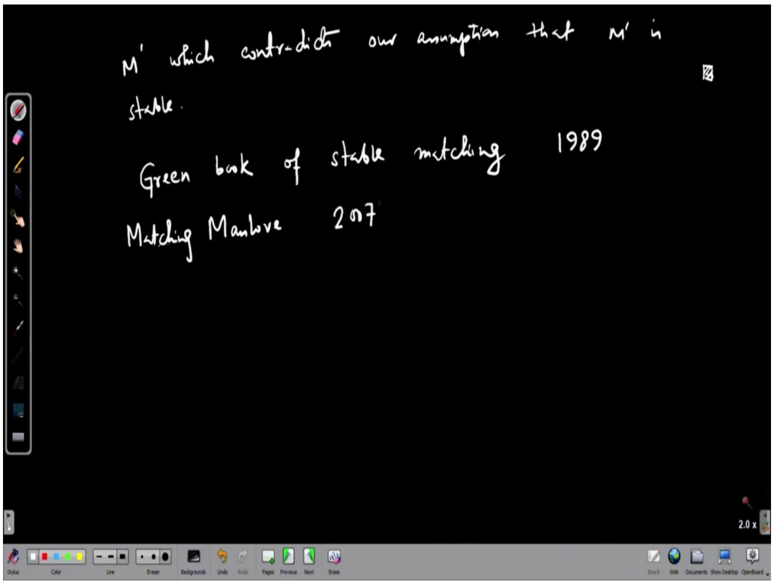
So, b is matched with someone here a double prime and in the $(i+1)$-th iteration it gets matched with it got a proposal from a and because she already is matched with some someone whom she prefers over a, she rejects proposal of a and look at a’s proposal again. So, if possible again prove by contradiction suppose there exists a stable matching M' containing (a,b) ok.

Now, you know you see that in that stable matching b is matched with a and a is matched with b. Now you ask that you know whom a prime is matched with? In a' b is somewhere here. Now you see that all the elements all the women whom a prime prefers over b they belongs to R_i because they have a prime has already proposed to all women who whom he prefers over b and there is no stable matching because of induct due to induction hypothesis where a' gets matched with b matched with those women.

But, a prime is also not matched with b. So, a prime must be matched with some woman after b whom he prefers less than b on the other hand b is matched with a. So, what we have is that you know if I look at a double primes preference a double primes this man a double prime prefers her partner his partner less than b and b also prefers her partner a which is $M'(b)$ then a'' prefers less.

So, here again (a'', b) forms a blocking pair for the matching M' which contradicts our assumption that M' is stable.

(Refer Slide Time: 27:50)



So, this finishes the proof and let me finish this is the last class let us finish with couple of remarks it is just a tip of iceberg this stable matching there is. This table matching is one of the most widely studied area in game theory algorithm mechanism design and I would strongly recommend you to read this Green book of stable matching it is called Green book of stable matching, which nicely summarizes all the findings till 1989.

And there is another book on stable matching which is called matching by Manlove I think it is called matching which is in 2007 which summarizes all the findings all the thing from 1989 to 2007, but it is still a very active area of research with this note let me end today's lecture.