Algorithmic Game Theory
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Lecture - 06
Indifferent Principle

Welcome to second week of our lectures. So, last week we stated a very important characterization of Mixed Strategy NASH Equilibrium without proof. So, today we will start with seeing the proof of that theorem.

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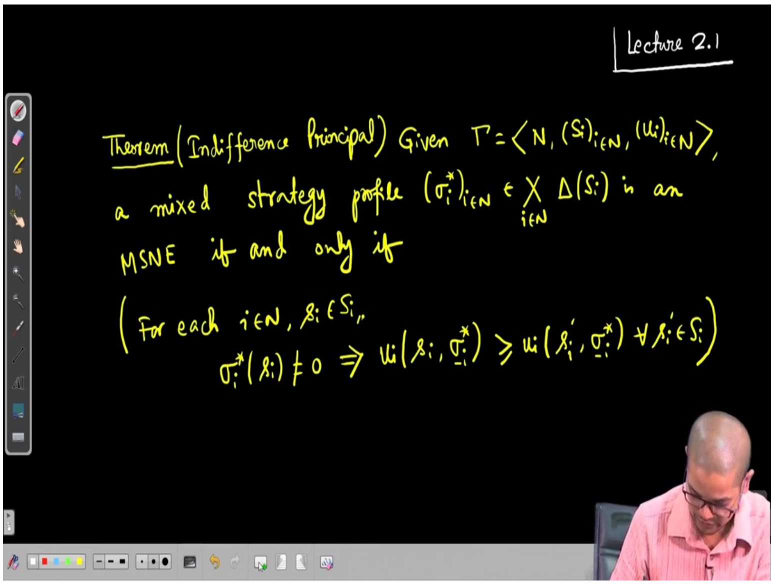

So, let me state what was the result let me state as a theorem, so this is called Indifference Principle Indifference Principle. So, what does it say? So, again as usual suppose here given a game gamma in normal form $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ a mixed strategy profile $(\sigma_i^*)_{i\in N} \in \prod \Delta(S_i)$ is an MSNE if and only if the following condition holds.

For each player $i\in N$ and its strategy $s_i \in S_i$, we have the following what we have that $\sigma_i^*(s_i)=0$ or not equal to 0 implies that the utility that player i derives by playing s_i. When other players continue to follow according to this mixed strategy profile sigma i star, this is at least as high as $u_i(s_i', \sigma_{-i}^*)$ this should hold for all $s_i' \in S_i$ value, so this entire condition.

So, sigma i star is a MSNE if and only if the entire condition within the bracket holds. What does it say? It says that for each player any strategy which gets non zero probability under σ_i^* should be a maximizer among the strategies available when other players are playing according to σ_{-i}^* ok.

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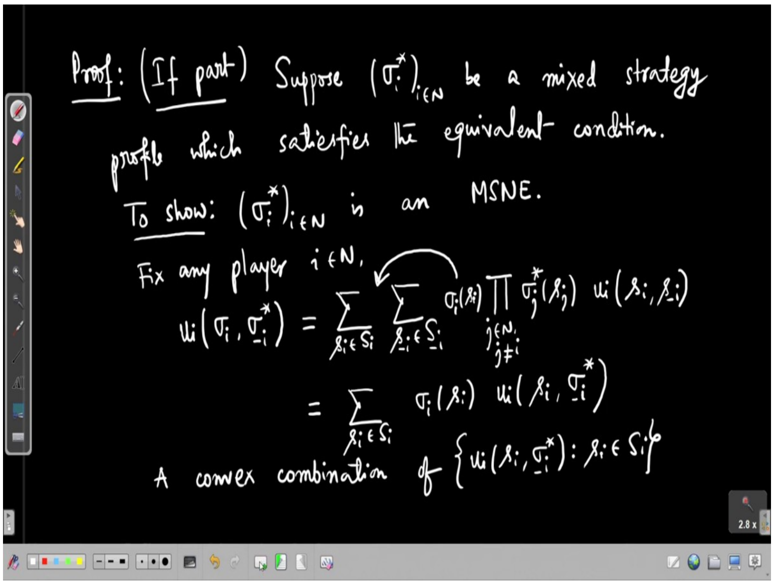

So, let us prove it proof. So, because it is an if and only if statement we need to show 2 parts first let show if part. So, suppose $(\sigma_i^*)_{i\in N}$ be a mixed strategy profile which satisfies the equivalent condition. What is the equivalent condition? That for each player i σ_i^* gives non zero probability only N only on those strategies which maximizes the utility of player i, when other players are playing according to the given mixed strategy profile.

So, suppose $(\sigma_i^*)_{i\in N}$ be such a mixed strategy profile what we need to show to show $(\sigma_i^*)_{i\in N}$ is an Mixed Strategy NASH Equilibrium MSNE. So, what is the definition of Mixed Strategy NASH Equilibrium? That unilateral deviation should not increase the utility of the player. So, fix any player fix any player $i \in N$, we will show that there is no benefit for player i to deviate from σ_i^* and play something else. Towards that let us write down what is the utility that player i is getting in this profile in this mixed strategy profile $(\sigma_i^*)_{i\in N}$, when it instead of playing σ_i^* it plays some other strategy any strategy σ_i.

When other players are playing according to the given mixed strategy profile, we need to show that by playing σ_i instead of σ_i^* player i cannot get more utility. So, let us expand it. Summation over $s_i \in S_i$ summation $s_{-i} \in S_{-i}$. Now what is the probability that this particular strategy profile (s_i, s_{-i}) is indeed played it is the probability of individual player playing this them playing their strategy.

So, I am just taking $\sigma_i(s_i)$ out and this is $\sigma_j(s_j), j \in N, j \neq i$, just the product probability. And what is the utility? $u_i(s_i, s_{-i})$. This is directly from the definition that definition of utility of a player i in a mixed strategy profile. So, what is it?

This what you do is that you take this $\sigma_i(s_i)$ further out of the inner summation $\sigma_i(s_i)$ and what is the remaining term then the remaining thing is utility of player i. When it plays s_i and the other players this is σ_{-i}^* are playing according to the given mixed strategy profile ok. Now, you see that this utility is a convex combination. So, this is let me highlight it, this is a this is a convex combination of this utilities this numbers ok. So, what is convex combination?

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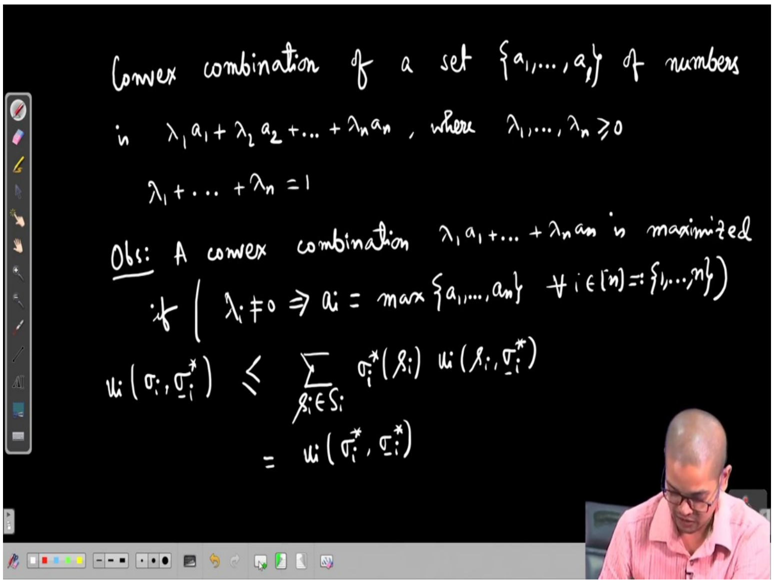


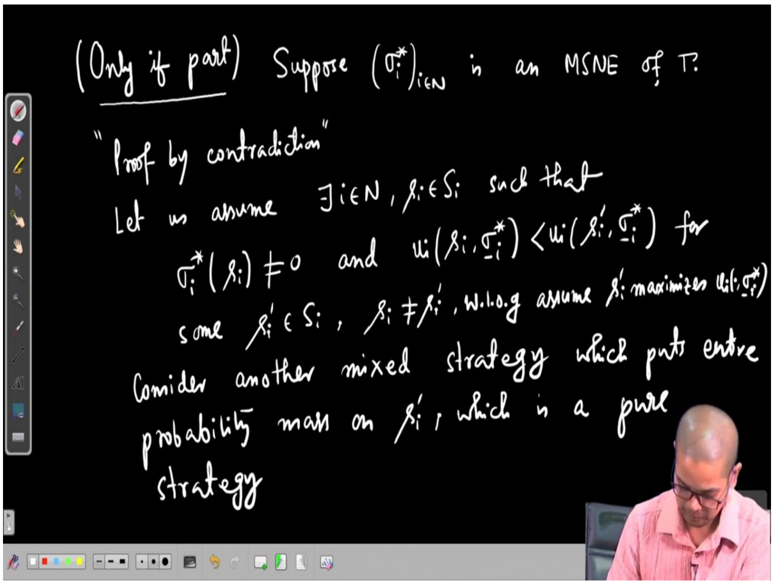
Let us let me explain what is convex combination? So, let me write convex combination of a set say $\{a_1, \ldots, a_n\}$ of numbers is a summation of the form $\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$.

Where all these lambdas $\lambda_1, \dots, \lambda_n$ these are greater than equal to 0 and they sum up to 1. Now an observation is that a convex combination like $\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n$ is maximized. If this coefficient that the non-zero coefficients are only on those highest terms if this condition holds. That means, $\lambda_i \neq 0$; that means, if a_i gets some non zero weightage then a_i must be maximum. This should hold for all i in n. What is this set? It is by definition the set of integers 1, 2 up to n.

So, this proof is very easy and you can prove it by induction i will leave it to you. So, continuing our proof from last page, we see that there the equivalent condition tells exactly like that that. Those σ are like lambdas here, the probabilities are like lambdas here and you know only those utilities which has maximum value gets nonzero weightage.

And hence what we get is $u_i(\sigma_i, \sigma^*_{-i})$ is maximized when we put all weightage only on maximum terms, which is exactly σ_i^* does $u_i(s_i, \sigma^*_{-i})$. Which is nothing but $u_i(\sigma_i^*, \sigma^*_{-i})$.

Hence by deviating from sigma i star to sigma i, player i cannot get more utility that this is exactly what we need to prove. So, this is one part this shows that if there exists a mixed strategy profile which satisfies this equivalent condition, then that must be a Mixed Strategy NASH Equilibrium. But we are claiming a characterization; that means, if i have a Mixed Strategy NASH Equilibrium that Mixed Strategy NASH Equilibrium will satisfies this equivalent condition.



So only if part, so always remember when we have a if and only if statement we need to prove 2 things on both directions we should prove. So, suppose I am given a mixed strategy profile which is a Mixed Strategy NASH Equilibrium. Suppose sigma i star i in N be an MSNE of the game gamma. What we need to show is that it satisfies the equivalent condition.

So, it is a proof by contradiction. So, if possible let us assume there exist a player $i \in N$ and the strategy of the player i $s_i \in S_i$ such that this particular strategy s_i gets nonzero probability under σ_i^* and it is not a maximizer $u_i(s_i, \sigma_{-i}^*)$ is strictly less than $u_i(s_i', \sigma_{-i}^*)$ for some $s_i' \in S_i$. Of course, I must have $s_i \neq s_i'$ then it cannot be strictly greater.

So, I claim that you know consider another mixed strategy, which so we can assume without loss of generality that si prime is a maximizer of this utility. So, assume without loss of generality without loss of generality assume s_i' maximizes the utility of player i when other players are playing according to σ_{-i}^*. So, just consider another mixed strategy which puts entire probability entire probability mass on s_i'. Just consider this particular mixed strategy which is a pure strategy right which is a pure strategy.

We will show that by playing s_i' is by playing si prime instead of σ_i^* player i is better off there by contradicting our assumption that $(\sigma_i^*)_{i \in N}$ is a MSNE.

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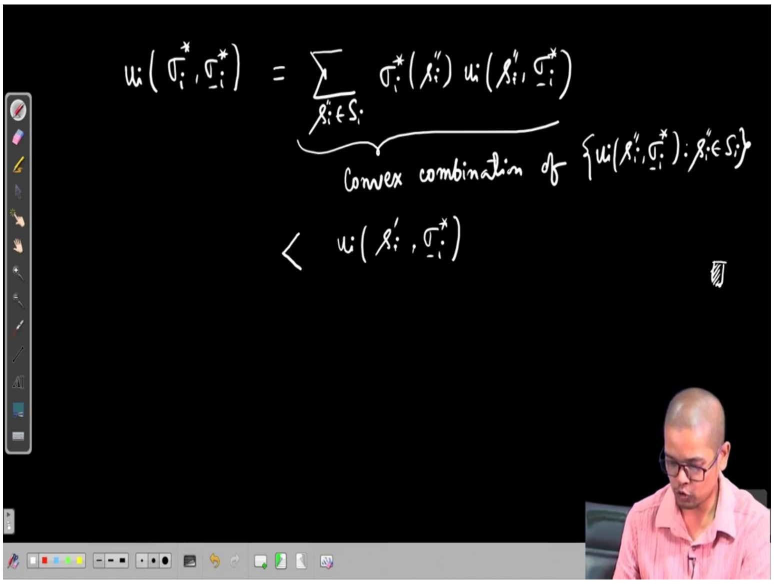

So, let us write what is utility of player i when it plays σ_i^* when other players are playing according to σ_{-i}^*.

So, as we have seen in the other direction we can write this as $s_i \in S_i$ let me use s_i'' $u_i(s_i', \sigma_{-i}^*)$. Now here again this is a convex combination this particular strategy this particular trick this particular idea will be used again and again, very simple but elegant and useful idea.

Convex combination of this utilities $u_i(s_i', \sigma_{-i}^*)$ $s_i' \in S_i$ and we know this will be maximum. If we put entire mass on the maximizer and that is an equivalent condition if there if we put any mass on any item other than maximizer then it is value will be strictly less. Now σ_i^* because $\sigma_i^*(s_i)$ is greater than 0 and the corresponding utility $u_i(s_i, \sigma_{-i}^*)$ is not maximum. We can write here this is strictly less than $u_i(s_i', \sigma_{-i}^*)$ ok, so which concludes the proof ok.

So, this concludes our study of our exposition of various equilibrium concepts. All these equilibrium concepts the main this is the; this is the this this answers. The second question can we predict what will happen, how will the players play and so on? So, let us list down what are the equilibrium concepts we have studied till now.

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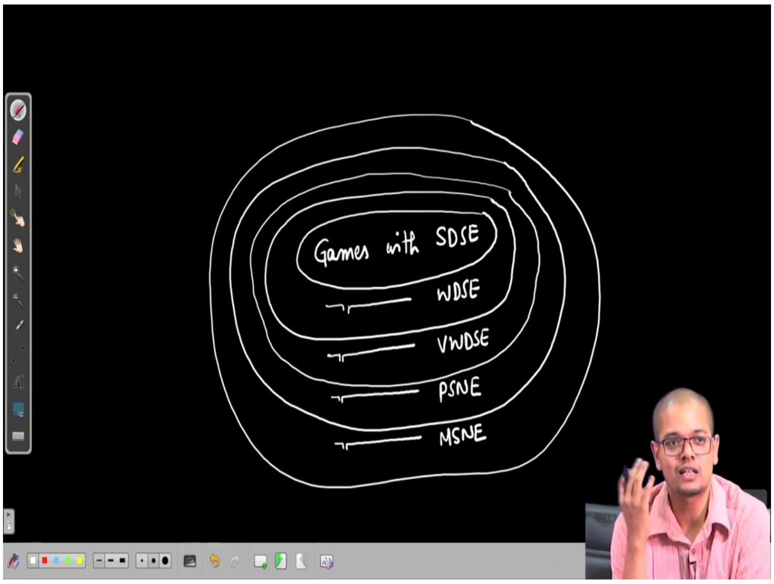

So, we began with strongly dominant strategy equilibrium and we have seen that there are they are there are games like say prisoners dilemma, which has a strongly dominant strategy equilibrium, games with strongly dominant strategy equilibrium.

We have also observed that all strongly dominant strategy equilibrium is also a weakly dominant strategy equilibrium, which is a weaker condition in that sense that there exist games which has a SDSE, but not weakly dominant strategy equilibrium but not strongly dominant strategy equilibrium. We have seen second price auction of this second price auction is an example of this kind of games.

This is games with weakly dominant strategy equilibrium. Then we have weakened the notion further to what is called very weakly dominant strategy equilibrium. It is easy to come up with examples of games which has very weakly dominant strategy equilibrium, but not weakly dominant strategy equilibrium.

So, weakened this notion of very weakly dominant strategy equilibrium and what we get is Pure Strategy NASH Equilibrium. Again we have observed we have seen games which has Pure Strategy NASH Equilibrium, but no very weakly dominant strategy equilibrium.

Examples of such game is, coordination games battle of success etc. But still there exist interesting games like matching pennies rock paper scissor which does not have a Pure

Strategy NASH Equilibrium and we weakened that notion further and we what we get is Mixed Strategy NASH Equilibrium.

And the celebrated NASH theorem states that all finite games have at least one Mixed Strategy NASH Equilibrium. So, from next class we will study a special very important and special class of games which are called Zero Sum Game also known as strictly competitive game or win loss game ok.

Thank you.