Algorithmic Game Theory
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Lecture - 59
Gale-Shapley Algorithm

Welcome. In the last class we have started studying mechanism design without money and as a model example we have started studying stable matching problem. And we have stated the Gale Shapley theorem and we will formally present that theory present algorithm deferred acceptance algorithm today and present its correctness.

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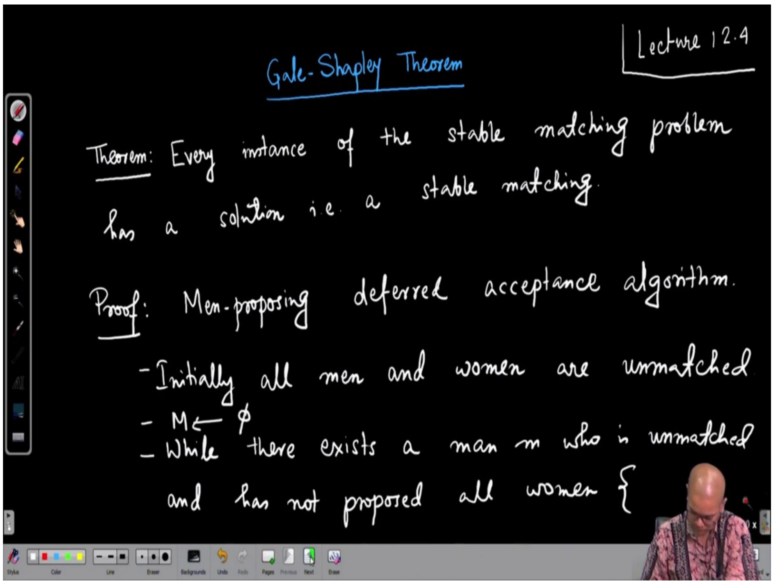

So, today's topic is Gale Shapley theorem. So, what is the theorem? Let us recall every instance of the stable matching problem has a solution what is the solution that is, stable matching. Proof: the proof is algorithmic we will present an algorithm which is called men-proposing deferred acceptance algorithm.

So, let us first describe the algorithm. In the last class we have described it intuitively and dry anterior dry run we show how this algorithm functions on a concrete example and show and showed how we get a stable matching. So, let us now little more formally explain the algorithm. So, initially all men and women are unmatched, then while there exists a man m who is unmatched and have no has not proposed to has not proposed all women while loop begins.

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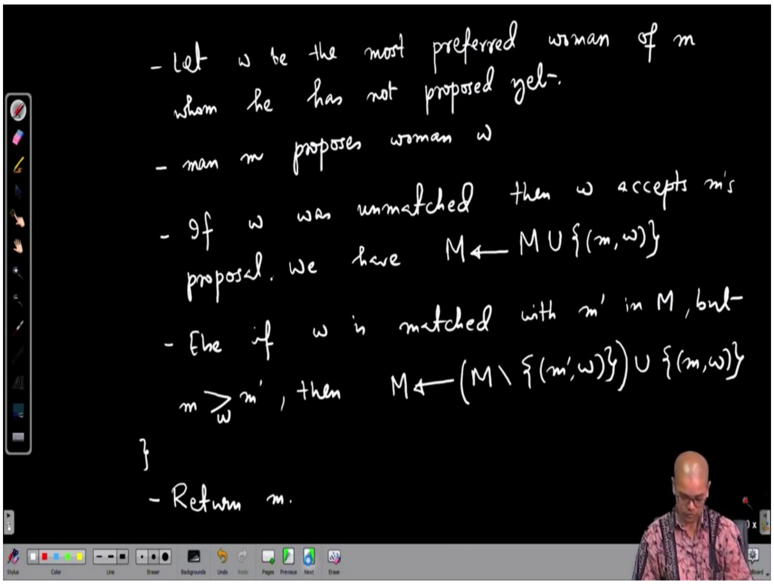

Let w be the most preferred woman of m whom he has not proposed yet. So, then m man m proposes woman w. If w was unmatched then w accepts m's proposal in this case we have the matching M we are maintaining which we are building iteratively and M was initialized to empty set. So, initialize M to empty set.

So, $M = M \cup (m, w)$ pair; else if w is matched with m prime in m, but w prefers m over m' then w rejects m prime and gets matched with M. So, we update $M = M \setminus (m', w) \cup (m, w)$. If this is not the case; that means, if w is matched with same prime and w prefers m' more than m.

Then w simply rejects m's proposal and there is no change of this matching m partial matching m the current iteration and so this continues so, if the while loop terminates then return m. Now so this is the description of the algorithm men proposing deferred acceptance algorithm.

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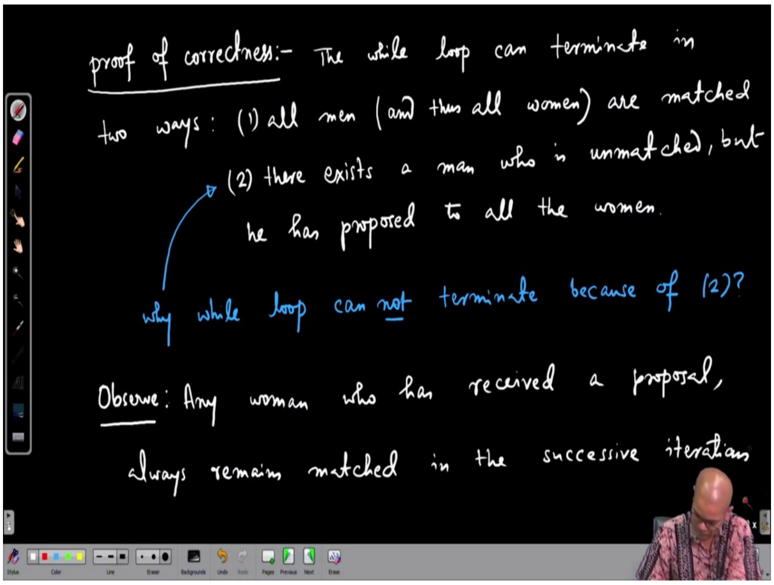

Now, we go to proof of correctness proof of correctness. So, first observe that while loop can terminate while loop can terminate in two ways. 1st way is all men and if all men are matched then all women are also matched. Because their numbers are same the number of men and number of women are same. All men and thus all women are matched.

2nd is that there exists a man who is or all unmatched men have proposed to all the women or there exists a man who is unmatched. But he has proposed to all the women first we show that the while loop cannot terminate in the second way. It will all it will always terminate in the first way when the while loop terminates all men and thus the all women must be matched. So, why while loop cannot target in second way? This is the question.

Why while loop cannot terminate because of the second condition why it cannot terminate? See first observe that any woman who has received a proposal always remains matched for the remaining of the algorithm for the for this or in the successive iterations of the algorithm matched in the successive iterations of the algorithm although she can change her partner.

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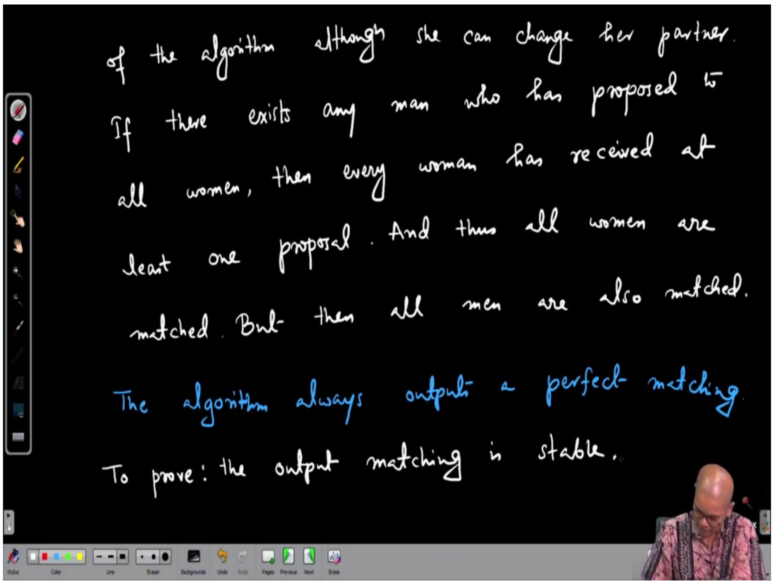

So, think of an unmatched woman whenever she receives a first proposal. She accepts that proposal because all men and women in particular that particular women prefers being matched compared to remain unmatched. Now, in subsequent iterations only those men who are unmatched their proposes and a women who is already matched. If he receives another proposal she can change her partner she can accept the new proposal if she likes the new proposal more than her existing partner.

So, she can change partner, but once match she remains matched this is not the case for the men a men may get matched in say ith iteration and later he can again become unmatched. Because the women he was matched she got better proposal and get matched with the better person. So, women once matched always remains unmatched you know always remains matched.

Now, if there exist any man who has proposed it is a proof by contradiction that you know the algorithm the while loop cannot terminate in the second way. So, suppose the algorithm the while loop terminates in the second way that there exist a man who has remained unmatched, but he has proposed to all the women so that is what we are saying.

If we have if there exists any man who has proposed to all women then all the women have received at least one proposal, then every woman has received at least one proposal. And thus all women are matched this implies, but then all men are also matched. But then the algorithm has already terminated the while loop has already terminated due to the fulfillment of fast condition that all men and all women are matched.

So, there cannot exist any unmatched man who has proposed to all the women. Because if any man has proposed to all the women at in the iteration when he is proposing to the to his last women all the women has received at least one proposal. That means all the women are matched and hence all the men are also matched.

So, there cannot exist any unmatched man who has proposed to all the women. So, what we get here is that the algorithm always terminates always outputs perfect matching a matching is called perfect if all the women and men are matched. Now what is not clear is that why this matching must be stable. Why there should not be any blocking pair?

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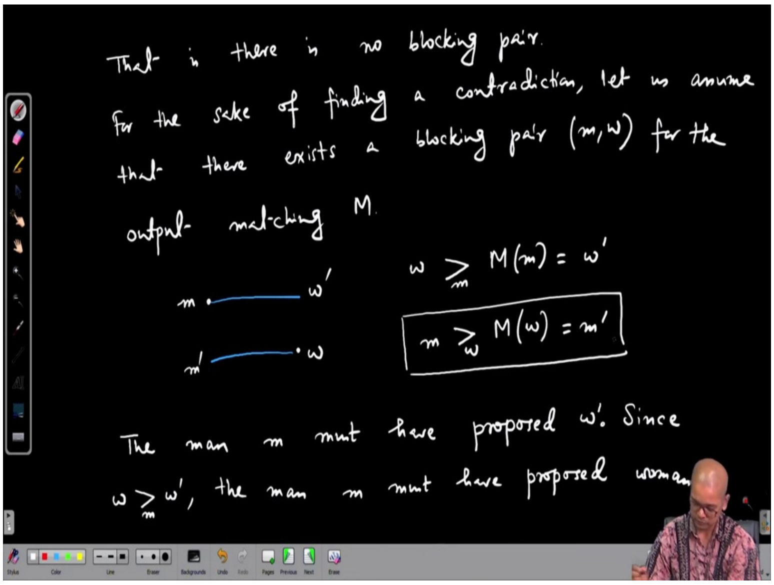

So, next to prove the output matching is stable that is there is no blocking pair that is what we need to prove. So, proof by contradiction for the sake of finding a contradiction. Let us assume that there exists a pair a blocking pair (m,w) for the output matching M.

Now, what does that mean? That means, that if I look at the matching then here is m this man m and here is this women w and they are not partners because if they are matched with each other under m. Then they cannot form a blocking pair then because m is a perfect matching the only way (m,w) from a ball blocking pair is that m prefers his partner say $w^{'}$ more than w and w also prefers her partner $m^{'}$ less than m.

So, m prefers his partner w' less than w. So, let us write m prefers w over his partner under this matching M which is w prime and w prefers m over her partner $M(w)$ which is m prime ok. Now, you see that this man m is matched with w' whom he prefers over w and what is the men proposing deferred acceptance algorithm. Each man keeps proposing women a women whom he has not proposed yet and whom he preferred most.

So, because m is eventually matched with w' m must have propose w prime. So, the man m must have proposed w prime. Now since this man prefers w over w' the man m must have proposed woman w in some iteration i.

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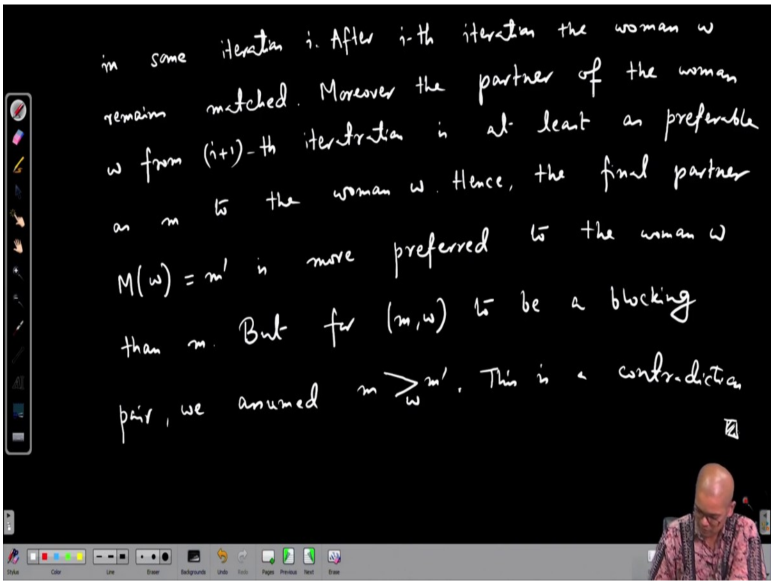

And you know then after ith iteration in the ith iteration man m propose w and. So, of from i after ith iteration the women w remains matched. Moreover the woman the partner of the woman w moreover the partner of the woman w from (i+1)-th iteration is at least as preferable as m to the woman w.

You see that in the ith iteration the women w has received a proposal from man m and at that state either the women was women w was matched and to some men whom she prefers less than m. Then in that case women w will receive will accept m’s proposal and get matched with m or women w is already matched with some men better than or whom he whom she prefers more than m.

And in the light in the later iterations also the woman can change her partner, but whenever she changes her partner he/she likes her new partner more. So, hence the final partner which is M of w equal to m prime is more preferred to the woman w than m. But this contradicts the assumption of blocking pair the assumption of blocking pair says that women w prefers m over m prime which is a contradiction.

But for m comma w to be a blocking pair, we assumed w prefers m over m prime this is the contradiction. So, the output matching is a perfect matching and it is a stable matching there cannot exist a blocking pair.

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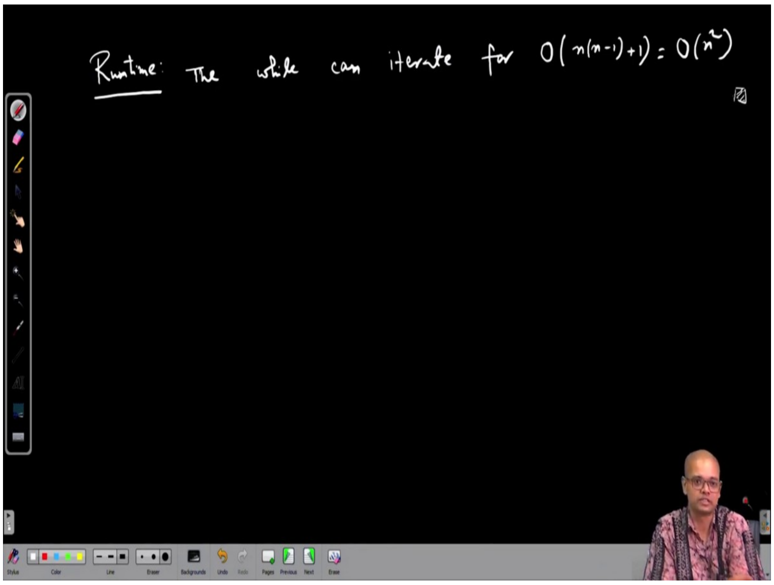

So, we conclude this by explaining what is the run time of this algorithm. Now how many times the while looped while loop can iterate the while loop can iterate for if there are n man and n women and each whenever there is a man who has proposed to all the women at that of at after that iteration all women must be matched.

So, the number of iterations can be at most n square. Because there are n men and each man can propose to at most n women at most (n-1) women actually. So, it is actually n(n-1)+1 which is $O(n^2)$. After n(n-1)+1 iteration there must exist at least one man who has proposed to all women.

And after that all men and all women must be matched. So, this iterates for $O(n^2)$ iterations. So, these are very efficient algorithm. So, we will stop here today in the next class we will study some further properties of stable matching.

Thank you.