Algorithmic Game Theory
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Lecture - 58
Stable Matching

Welcome. So, till now we have studied mechanism design and that comes under the scenario which is called a mechanism design without money the quasi linear environment.

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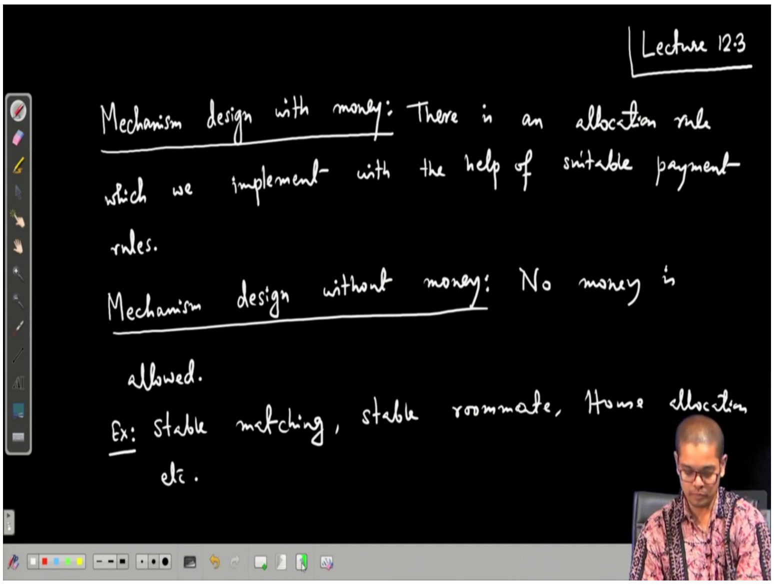

So, let us write. So far whatever you have studied under quasi linear environment they are called mechanism design with money mechanism design with money and what is the high level structure that, there is an allocation rule there is an allocation rule and which we implement with the help of suitable payment rules.

We incentivize players using money to reveal their true type. So, that the social planner can implement the social choice function that he or she wants to implement or the allocation rule which he or she wants to implement. There is another vast field of mechanism design which is called mechanism design without money mechanism design without money, here no monetary transaction is allowed no money is allowed, examples of such mechanisms.

So, all what you have to do you have to you have the algorithm and using that you need to study you need to try to achieve dominant strategy incentive compatibility and so on. So, what are the some of the examples say one example is Stable Matching or stable roommate, house allocation etcetera. So, we begin our study of mechanism design without money using stable matching.

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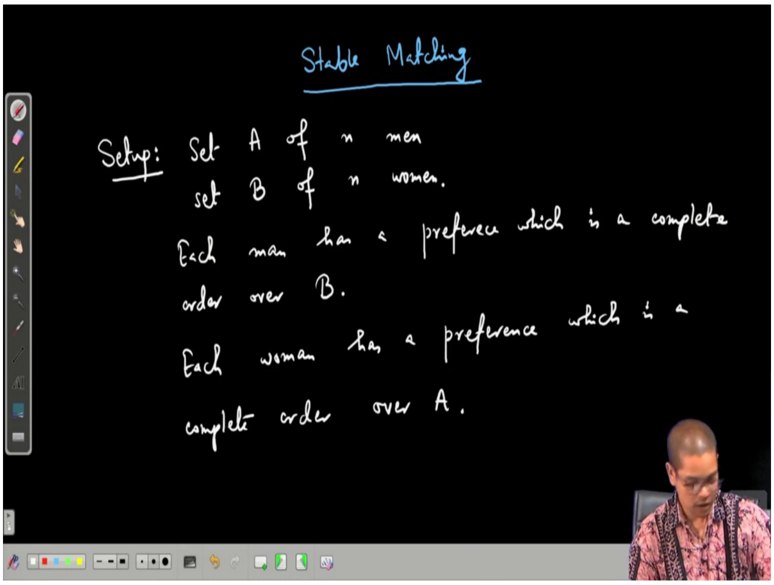

So, what is the setup? We have a set A of n men, set B of n women, each man has a preference order preference which is a complete order over B. The set of all women similarly each woman has a preference which is a complete order over the set of all men A. And, what we are looking for?

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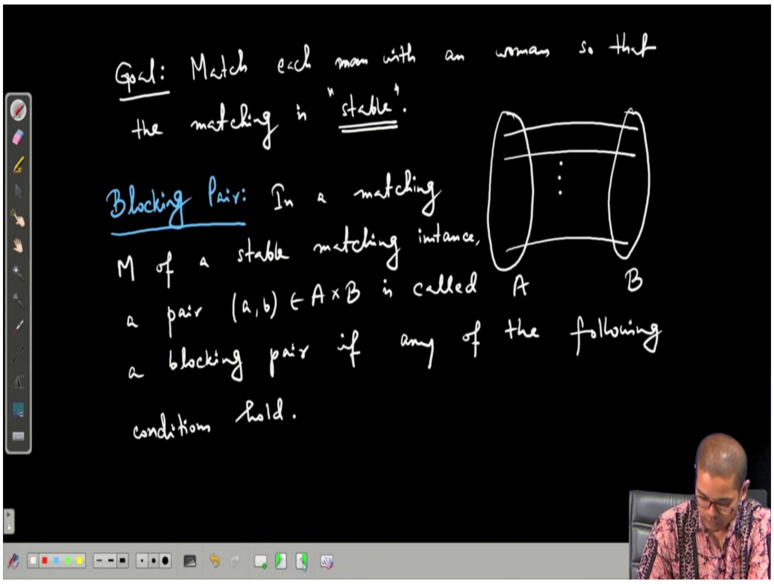

So, goal is to match each man with an woman so that the matching is "stable." So, what do what do you mean by matching? So, suppose this is the set of men A, this is the set of men B and we are matching each man with a woman there are n men and n women. So, for each man there is a woman and for each woman there is a man they are they are partners.

So, what is stable matching? What is this what do you mean by this stable? To define stable we define a concept called blocking pair so, blocking pair. So, in a matching M matching M is nothing but a collection of it is a collection of pairs men women pairs. In a matching M of a stable matching instance a pair $(a,b) \in A \times B$ a pair of men and women is called a blocking pair if any of the following holds any of the following conditions hold.

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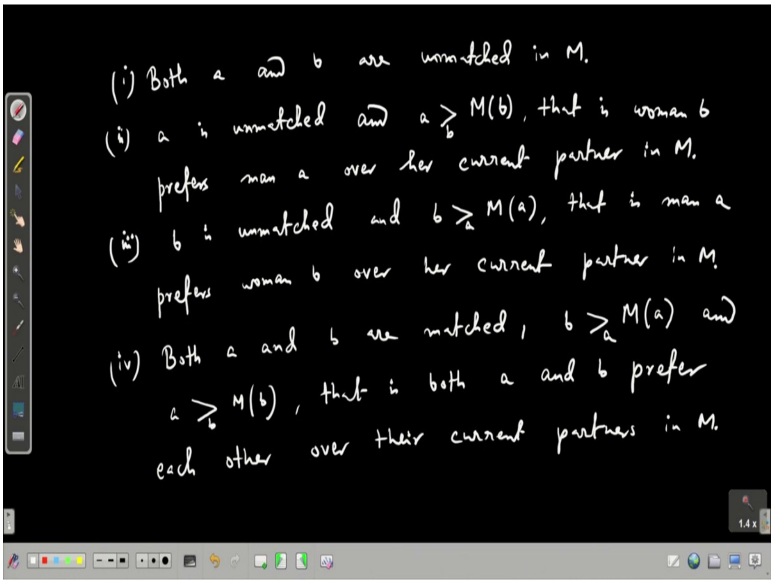

The first condition is that, both a and b are unmatched in M. So, a matching M is a set of pairs and it leaves both a and b. So, each man prefers each man and each woman prefer getting matched compared to remaining unmatched. So, if there is a matching M which leaves both a and b then this matching is unstable because you know then this at least this pair of men and women a and b would prefer to get matched compared to remain unmatched in M. So, unmatched in M.

Second is that, a is unmatched and b prefers its current partner less than a b prefers a more than its current partner which we denote by $M(b)$ this is b. So, by this notation that is let me write that is woman b prefers her prefer a prefer man a over her current partner in M, which is $M(b)$.

Then in this case also the woman b would be happy to leave her current partner and get matched with a. So, then then this matching M is not stable there is a pair of men and women who tries to disrupt M and change the matching in that sense this is not stable, this is a blocking pair, this pair blocks this matching M to be stable.

Then the symmetric condition b is unmatched and man a prefers b over its current partner that is man a prefers woman b over her current partner in M. And the fourth condition is that, both a and b are matched a and b are matched, but a prefers b over his current partner and b prefers a over her current partner.

In this case also both a and b would be happy to leave their current partners and get matched themselves. So, that is both a and b prefer each other each other over their current partners in M. So, these are called blocking pairs.

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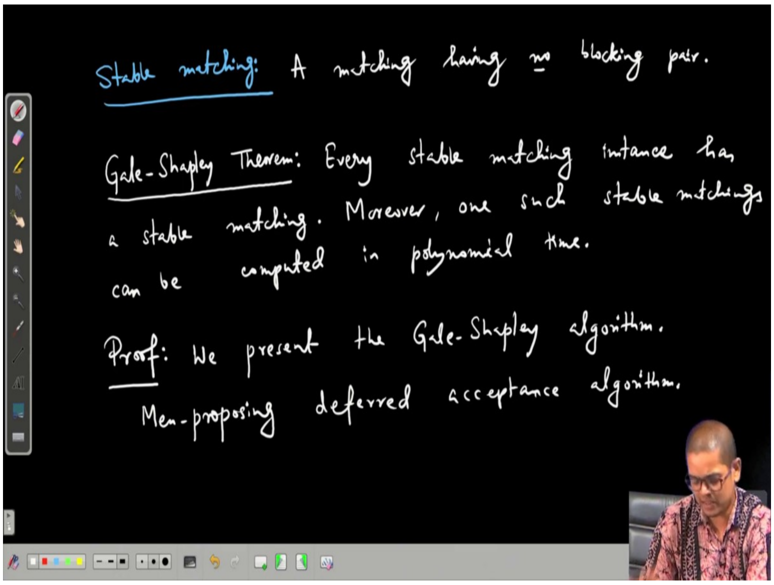

And what is a stable matching? So, a stable matching is a matching having no blocking pair this is what is a stable matching, matching which does not have any blocking pair. Now, interesting question is a natural question is that, does every instance of stable matching has a stable matching? And if it is yes, then can it be computed in polynomial time. And there is a famous theorem of Gale and Shapley due to which primarily due to which they are awarded Nobel Prize is Gale and Shapley theorem.

Gale-Shapley theorem, it states that every stable matching instance, what is a stable matching instance? There is a set of n men n women for each man there is a preference order which is a complete order over all the women and for each women there is a complete order over the men that is what a stable matching instance is.

And every stable matching instance has a stable matching. Moreover, one such stable matchings stable matching may not be unique, but one of them at least one of them can be computed in polynomial time can be computed in polynomial time. Proof; the proof is very algorithmic and so, we present the Gale - Shapley algorithm the Gale-Shapley algorithm. So, what is the Gale-Shapley algorithm?

So, there are various versions there are two versions, the version that we will propose we will present is called men proposing Gale-Shapley algorithm men proposing sorry deferred acceptance algorithm, men proposing deferred acceptance algorithm. There is another one which is women proposing deferred acceptance algorithm which is symmetric. So, let us understand this men proposing deferred acceptance algorithm let us understand this with an example.

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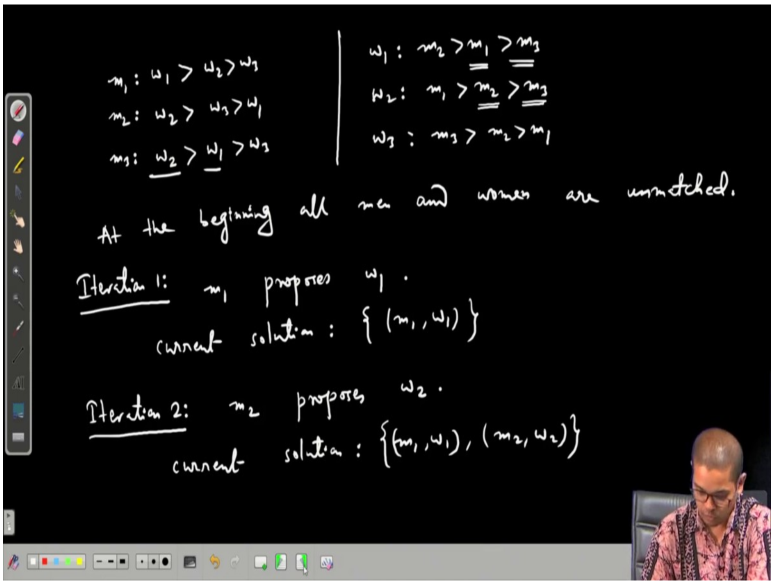

So, suppose we have 3 men and 3 women say m_1, m_2 and m_3 and their preferences be let us write something arbitrary, suppose m_1 prefers w_1 over w_2 over w_3, suppose m_2 prefers w_2 over w_3 over w_1, suppose m_3 prefers w_2 over w_1 over w_3 and for women suppose w_1 prefers m_2 over m_1 over m_3, w_2 prefer say m_1 over m_2 over m_3 and suppose w_3 prefers m_3 over m_2 over m_1 ok.

So, how does the algorithm proceeds? Initially, all men and women are unmatched. So, at the beginning all men and women are unmatched and we pick any unmatched men because it is a man proposing or men proposing algorithm pick any unmatched men there are all of them are unmatched. So, let us pick m_1, let m_1 propose the most preferred woman whom m_1 has not proposed yet. So, in the beginning m_1 has not proposed any woman and m_1 likes w_1 most. So, m_1 proposes to w_1.

So, this is iteration 1, it is a iterative algorithm iteration 1, what happens m_1 proposes w_1 ? Now, the woman who receives a proposal if she is unmatched she accepts the proposal because simply because every man and woman prefer being matched than unmatched. So, the current solution is current solution which we will we can change we will change iteratively is m_1 is matched with w_1.

In the second iteration, let us see what happens, again I pick in an unmatched man and m_1 is matched m_2 and m_3 are still unmatched. So, let us pick m_2 and m_2 proposes to w_2. Again because w_2 is unmatched w_2 accepts the proposal of m_2. So, the current solution is m_1 matched with w_1 and m_2 matched with w_2. So, in the next iteration what happens, again I pick an unmatched man and m_3 is the only man unmatched.

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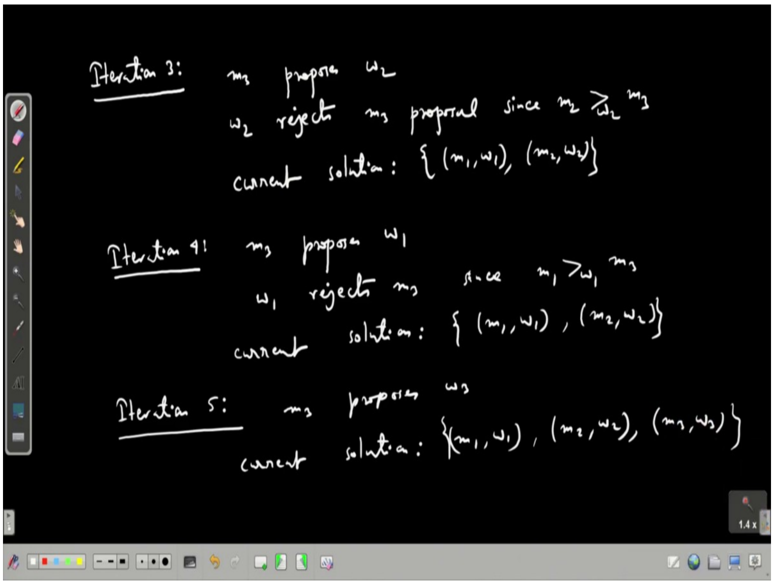

So, iteration 3, m_3 proposes w_2 because w_2 is the is the most preferred woman for m_3. Now, you see that m_3 is w_2 is currently matched with m_2 and she receives a proposal from m_3 whom he likes less than her current partner. So, w_2 rejects m_3s proposal.

So, w_2 rejects m 3s proposal since w_2 prefers her current partner m_2 over m_3. If it was otherwise if w_2 preferred m_3 over m_2 then w_2 would have accepted m_3s proposal and rejected her current partner m_2 in that case m_2 would have become unmatched again. So, after third iteration itself the after third iteration also the current solution remains same.

What is the current solution, m_1 matched with w_1 and m_2 matched with w_2. Let us see what happens in the next iteration.

Iteration 4, m_3 proposes m_3 has already proposes w_2 now it proposes w_1, now w_1 is currently matched with m_1 and it prefers m_3 less. So, it rejects m_3's proposal also. So, m_3 proposes w_1 and, but w_1 rejects m_3 since w_1 prefers m_1 over m_3. So, again current solution remains same, m_1 matched with w_1, m_2 matched with w_2. Iteration 5, next m_3 proposes w_3 and w_3 is unmatched and w_3 prefers getting matched over remaining unmatched.

So, the solution the current solution is m_1 matched with w_1, m_2 matched with w_2 and m_3 matched with w_3. You see that at this stage there is no unmatched man and the algorithm stops here with this matching as output and the claim is that this is a stable matching, why this is so, why the algorithm always terminates with the matching and why the matching is a stable matching, all these things we will see and we will formally prove it in the next class ok.

Thank you.