Algorithmic Game Theory
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Lecture - 56
Intermediate Domain

Welcome, in the last week we have being saying the set of all implementable functions in the single parameter domain and in the last class in particular we have seen a concrete example of sponsored search auction. So, let us briefly recall what we have been doing till now.

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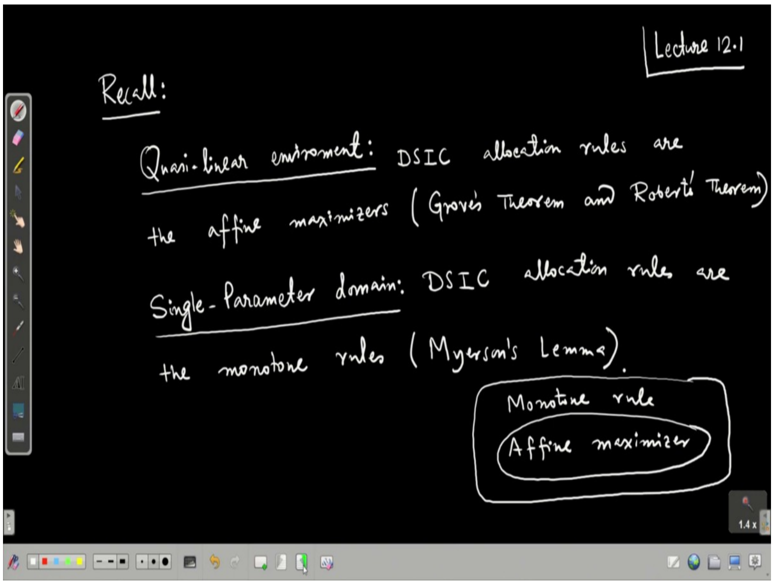

We have seen that in the quasi linear environment without any assumption, quasi linear environment in the quasi linear environment dominant strategy incentive compatible allocation rules are the affine maximizers. This is due to Grove's theorem and Robert's theorem.

This is the; this is the quasi linear environment in its full generality the most restrictive one or which is single parameter domain which is a subset of quasi linear environment where the type is parameterized by one real number single parameter domain. Here the dominant strategy incentive compatible allocation rules are by the way by when we say an allocation rule is dominant strategy incentive compatible.

We mean that there exist a suitable payment rule which makes this social choice function dominant strategy incentive compatible. Here the allocation rules that are dominant strategy incentive compatible are the monotone rules ok and this is due to Myerson's lemma and so monotone rules forms a superset strict superset of affine maximizers.

So, if here are affine maximizers then monotone allocation rules monotone allocation rules are strict superset of affine maximizers. In particular we have seen examples of monotone allocation rules which are not affine maximizer.

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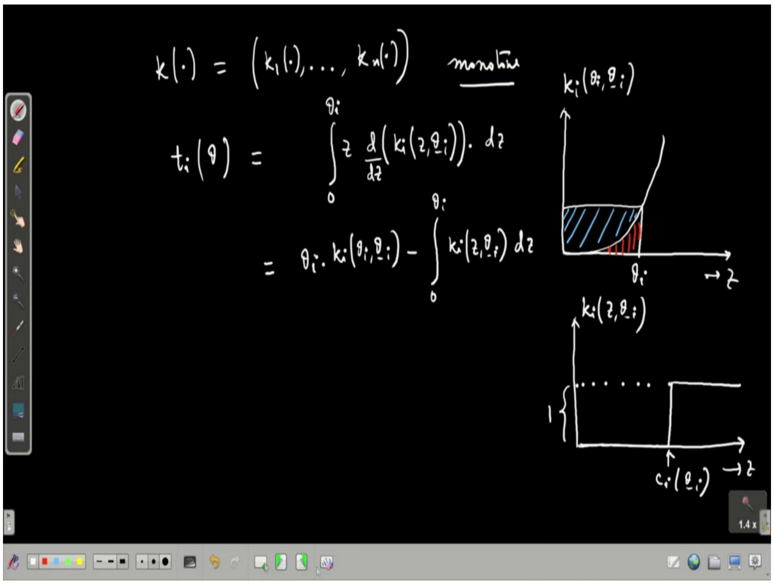

And then we studied Myerson's lemma which is very important. It basically says that if I have an allocation rule $k(.)=(k_1(.),\ldots,k_n(.))$. And if this is monotone; that means, $k_1(.),k_2(.),\ldots,k_n(.)$ is monotone and let us recall what is monotonicity means, monotonicity means that if the type profile of other players are fixed and if a particular player wins at a type say θ_i and if player i continues to increase its type then the player i continues to win.

It should not happen that because player i is value type increases if other players type profile are fixed then player i will continue to win that is what is called monotone monotonicity. And if it is monotone then the payment $t_i(\theta)=\int_0^{\theta_i} z\frac{d}{dz}k_i(z,\theta_{-i})dz$. And pictorially how does it look? Here is θ_i player is type and here is allocation $k_i(\theta_i, \theta_{-i})$ keep θ_{-i} fixed and if this looks like this.

Then because k_i must be monotone; that means, it must be a non decreasing function then at some value say at theta i. So, let us call this z, the payment is given by the area of this region. So, this formula can be alternatively written as $\theta_i k_i(\theta_i, \theta_{-i}) - \int_0^{\theta_i} k_i(z, \theta_{-i}) dz$ these two are same.

So, second term is the area of this red region. So, from this rectangle if I subtract the area of this vector red region I get the area of blue region. And also notice that this matches with our specialized special case of the single parameter domain which we started in the beginning that you know the each player has a two type of outcome whether player wins or loses.

So, in that particular case this allocation function looks like a step function. So, it remains there for some point and then it goes there and it takes only two values 0 and 1. So, this is 1, this is z, this is $k_i(z, \theta_{-i})$ and this particular point is exactly what was we called critical bid $c_i(\theta_{-i})$ ok.

So, this characterizes the set of all allocation rules which are implementable in dominant strategy equilibrium in two extremes. When we do not have any conditions; that means, we only assume quasi linear environment. Then the allocation rules implementable and demonstrated equilibrium or the affine maximizer.

And if in the other extreme if the types can be expressed as one real number that is single parameter domain then we have monotone allocation rules. What about some what about domains in between? There are some partial characterizations are known and that let us study now.

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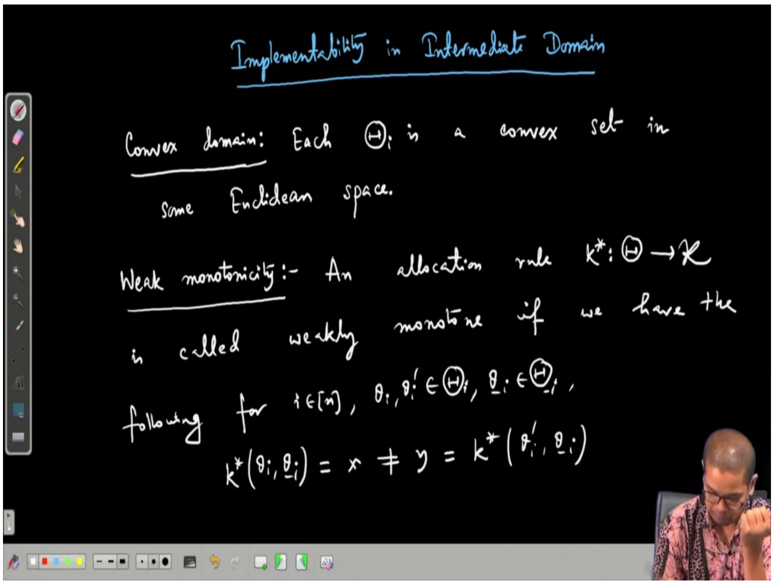

So, implementability in intermediate domain. So, we show we will see partial characterization for convex domains, convex domain. What is a convex domain? Each theta i is a convex set in some Euclidean space ok. And we will see that you know weak there is there is some concept of monotonicity which is called weak monotonicity which sort of characterizes the allocation rules which are implementable in domain strategy equilibrium in convex domains.

So, what is weak monotonicity? So, an allocation rule k star from theta to k is called weakly monotone if we have the following for every player $i \in [n]$; two types $\theta_i, \theta_i^{'} \in \Theta_i$ type profile of other players $\theta_{-i} \in \Theta_{-i}$. And the allocations are different allocations in (θ_i, θ_{-i}) and $(\theta_i^{'}, \theta_{-i})$ they are different.

So, $k^*(\theta_i, \theta_{-i})$ suppose this is x and this is different from y which is the allocation at $(\theta_i^{'}, \theta_{-i})$. So, I take two type profiles where the type profile of other players remain same θ_{-i} only the type of player i varies and then the outcome also changes from x to y.

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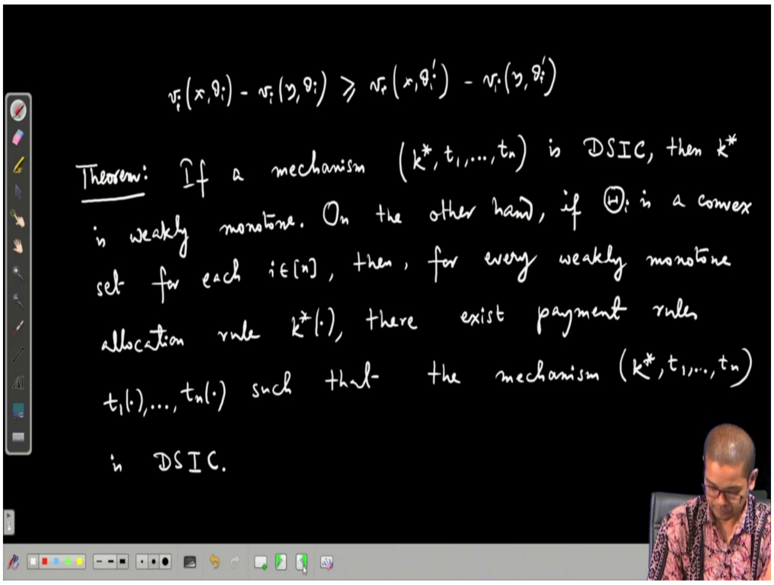

If such a thing happens then the following condition should hold $v_i(x,\theta_i) - v_i(y,\theta_i) \geq v_i(x,\theta_i^{'}) - v_i(y,\theta_i^{'})$. So, let us prove this. So, this is the this is called weak monotonicity where x is the outcome chosen at type profile (θ_i, θ_{-i}) it is an allocation chosen and y is the allocation chosen at the type profile $(\theta_i^{'}, \theta_{-i})$.

So, theorem if a mechanism $(k^*(.), t_1(.), \ldots, t_n(.))$ is dominant strategy incentive compatible then $k^*(.)$ is weakly monotone. On the other hand; on the other hand if theta i is a convex set for each $i \in [n]$. Then for every weakly monotone allocation rule $k^*(.)$ there exists payment rules t 1 to t n; such that the mechanism $(k^*(.), t_1(.), \ldots, t_n(.))$ is dominant strategy incentive compatible.

So, this theorem characterizes the set of all dominant strategy incentive compatible allocation rules in a convex domain. So, let us prove it.

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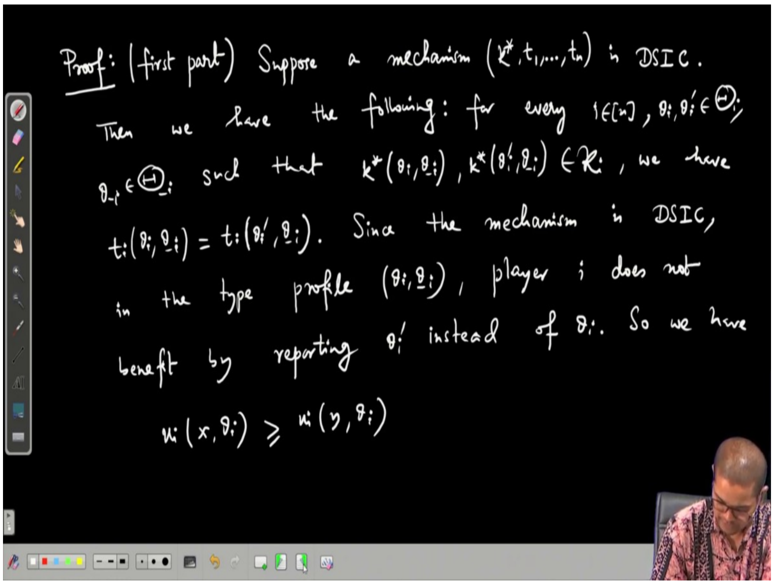

Proof: so, it has two part first part is if the given mechanism is dominant strategic incentive compatible then we need to show that the allocation rule is weakly monotone so first part. So, suppose a mechanism $(k^*(.), t_1(.), \ldots, t_n(.))$ is dominant strategy incentive compatibility ok.

So, then we have seen then we have the following; then we have the following for every $i \in [n], \theta_i, \theta_i^{'} \in \Theta_i$ types of player i $\theta_{-i} \in \Theta_{-i}$ type profile of other players. Such that $k^*(\theta_i, \theta_{-i})$ and $k^*(\theta_i^{'}, \theta_{-i})$ both belongs to K_i. We have the payment should be same (θ_i, θ_{-i}).

If the type profile of other players do not change and if the outcome depends belongs to K_i then the out then the payment also remains same. Because the payment depends on θ_i only via the allocation $t_i(\theta_i^{'}, \theta_{-i})$. Now since the mechanism is DSIC in the type profile (θ_i, θ_{-i}) player i does not benefit by reporting $\theta_i^{'}$ instead of θ_i.

So, we have utility of player i $u_i(x, \theta_i) \geq u_i(y, \theta_i)$ and utility is valuation plus payment.

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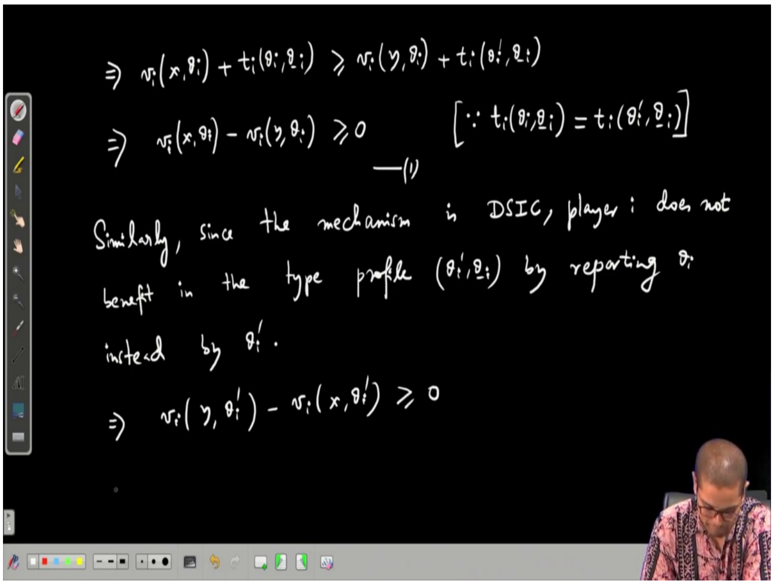

So, this is $v_i(x,\theta_i)+t_i(\theta_i,\theta_{-i})\geq v_i(y,\theta_i)+t_i(\theta_i^{'},\theta_{-i})$. Now because $t_i(\theta_i,\theta_{-i})=t_i(\theta_i^{'},\theta_{-i})$, we have $v_i(x,\theta_i)-v_i(y,\theta_i)\geq 0$. Since the payments are same $t_i(\theta_i,\theta_{-i})=t_i(\theta_i^{'},\theta_{-i})$ so, this equation 1. Now what we do is that we apply the same principle for the other profile.

So, similarly since the mechanism is dominant strategy incentive compatible, player i does not benefit in the type profile $(\theta_i^{'},\theta_{-i})$ by reporting θ_i instead of $\theta_i^{'}$. So, again the same line of argument will show that $v_i(y,\theta_i^{'})-v_i(x,\theta_i^{'})\geq 0$.

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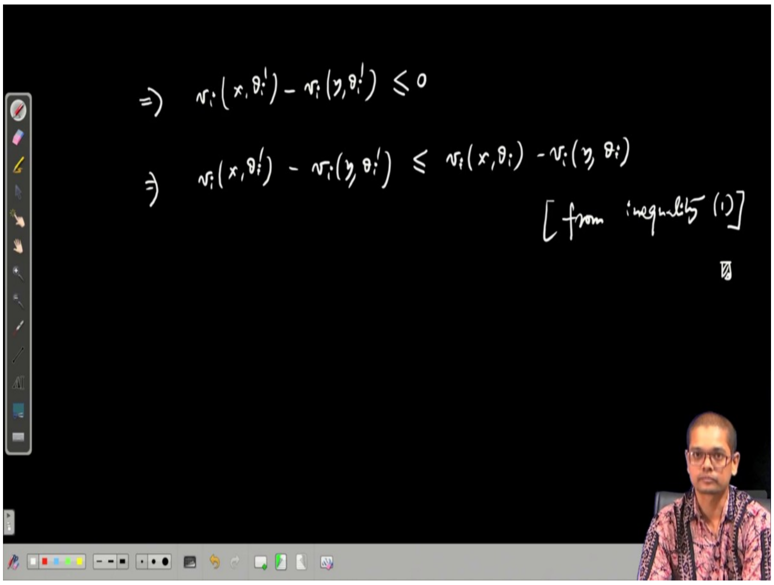

Now, shifting all the terms in the other side we get that $v_i(x, \theta_i^{'}) - v_i(y, \theta_i^{'}) \leq 0$. But from equation 1 we have $v_i(x, \theta_i) - v_i(y, \theta_i) \geq v_i(x, \theta_i^{'}) - v_i(y, \theta_i^{'})$ this is from equation 1 from inequality 1 from ok. So, you show partial characterization that the other direction is more complex and that is out of scope of this course. But again so for this convex domain this weak monotonicity characterizes the set of allocation rules. So, we will stop here today.

Thank you.