Algorithmic Game Theory
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Lecture - 54
Mayerson's Lemma

Welcome, so from last couple of lectures we have been studying single parameter domain and we will continue that in this lecture also.

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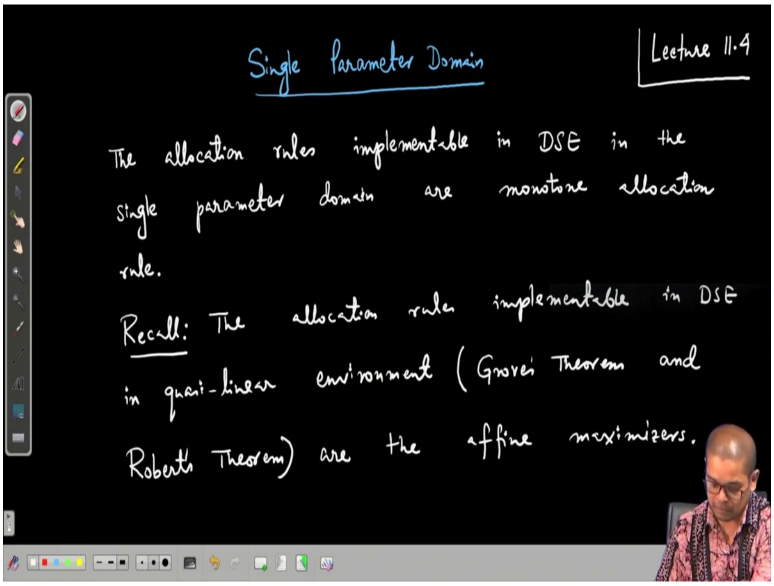

So, let us start Single Parameter Domain. So, in the last class you have seen a characterization that the allocation rules must be monotone allocation rule, the allocation rules implementable in dominant strategy equilibrium in the single parameter domain are monotone allocation rules.

And you know we have seen that affine maximizers recall: The allocation rules implementable in dominant strategy equilibrium in quasi linear environment. Again let us recall what are the theorems Groves Theorem generalized Groves theorem which says that affine maximizers are the implementable if such an allocation rule is affine maximizer. Then there exists a corresponding payment rule which makes it implementable in dominant strategy equilibrium.

And Robert's theorem says that essentially if you have at least 3 outcomes, then these affine maximizers are the only rules which are implementable Groves theorem and Robert's theorem. The allocation rules implementable in dominant strategy equilibrium in quasi linear environment are the affine maximizers; now, because, single parameter domain is a special case of quasi linear environment which has more structure.

Of course, we will expect more social more allocation rules to be implementable in particular all the allocation rules which are implementable in quasi linear environment namely affine maximizers, they should continue to be implementable in single parameter domain.

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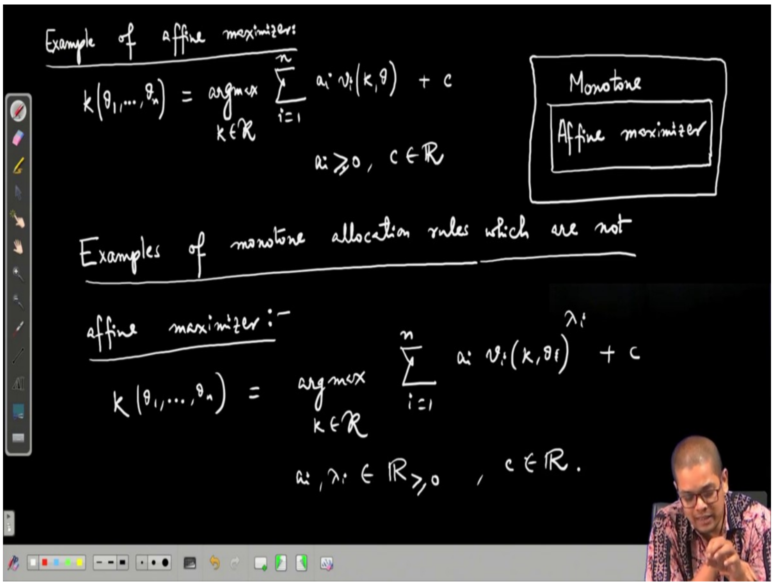

And that is why this diagram is affine maximizer, this is the set of all affine maximizers this is a subset of monotone allocation rules. Monotone allocation rules ok in particular each affine maximizer is a monotone allocation rule. Let us recall how does the affine maximizer look like $k(\theta_1,\ldots,\theta_n)$ should be arg max of the allocations $k \in K$; such that which maximizes an affine function $\sum_{i=1}^{n} a_i v_i(k,\theta)+c$ where this a_i s are constants a_i s are greater than equal to 0 and c is any arbitrary number positive or negative.

Now let us see some example of a monotone allocation rule which are not affine maximizer right, examples of so this is examples of affine maximizer.

Example of monotone allocation rule, monotone allocation rule which is not affine maximizer. Of course, all affine maximizers are also monotone allocation rule you can check if you are not convinced examples of monotone allocation rules which are not affine maximizers.

Say this for example, $k(\theta_1,\ldots,\theta_n)$ this is $arg\,max_{k\in K}\sum_{i=1}^{n} a_i v_i(k,\theta_i)^{\lambda_i}+c$. Where this constants a_i and λ_i is these are non negative real numbers and this constant c is can be both positive and negative.

So, these are examples of monotone allocation rules which are not affine maximizer, these allocation rules will not be implementable which are not affine maximizers which maximizers which this cannot be implementable in arbitrary quasi linear environment. This can be implementable this can be implemented in dominant strategy dominant strategy equilibrium only in single parameter domain.

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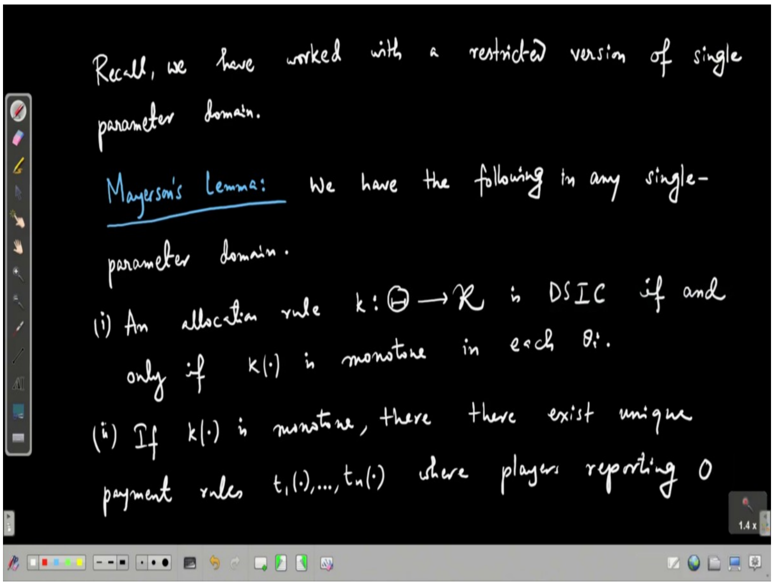

Now, recall that we have assumed because we have assumed some special structure of single parameter domain like you know the allocation rules, from each players perspective can have only two parts either winning or losing and the and so on and so on. But you know really speaking the single parameter domain means that the type of each player is a scalar quantity is a just one real number say that is it.

So, whatever we have studied in single parameter domain till now that holds true only in the only in that very special kind of single parameter domain the way we defined. Now what we will do is that we will relax that and with the understanding that single parameter domain means that each players type is just a single one real number which is it is type and in that most general single parameter domain what are the set of all allocation rules which are implementable.

So, let us write that recall we have worked with a restricted version of single parameter domain, it is a very restricted version. So, what is what are the implementable social choice implementable allocation rules, which are implementable in dominant strategy equilibrium in single parameter domain unit it is full generality and that is the famous Myerson's lemma primarily because of which he has been awarded the Nobel Prize Myerson's lemma.

So, let us take the Lemma we will not prove it. So, we have the following in any single parameter domain. An allocation rule k from theta to k is DSIC dominant strategy incentive compatible. What do we mean by an allocation rule to be DSIC it means that there exists a corresponding payment structure, which when combined with this allocation rule k will get a social choice function if and that particular social choice function is dominant strategy incentive compatible.

So, an allocation rule is dominant strategy incentive compatible if and only if k is monotone in each θ_i in each component it is monotone. So, as you see that although the we had a restricted definition of single parameter domain, of course what were implementable in that restricted single parameter domain they will continue to be implementable in an arbitrary single parameter domain.

But sorry the other way that if it says that the first part says that the set of all implementable allocation rules remain exactly same. So, the restrictions that we have put on our definition of single parameter domain are not that serious that is what it says. So, in this in the sense that in both the cases in both in full generality full general arbitrary single parameter domain and the restricted single parameter domain the set of all allocation rules which are implementable remains same they are monotone allocation rules.

2nd part if k is monotone then there exists unique payment rules $(t_1(.),\ldots,t_n(.))$, where you know in this most arbitrary single parameter domain there is no concept of player winning and losing they can be fractionally winning and so on.

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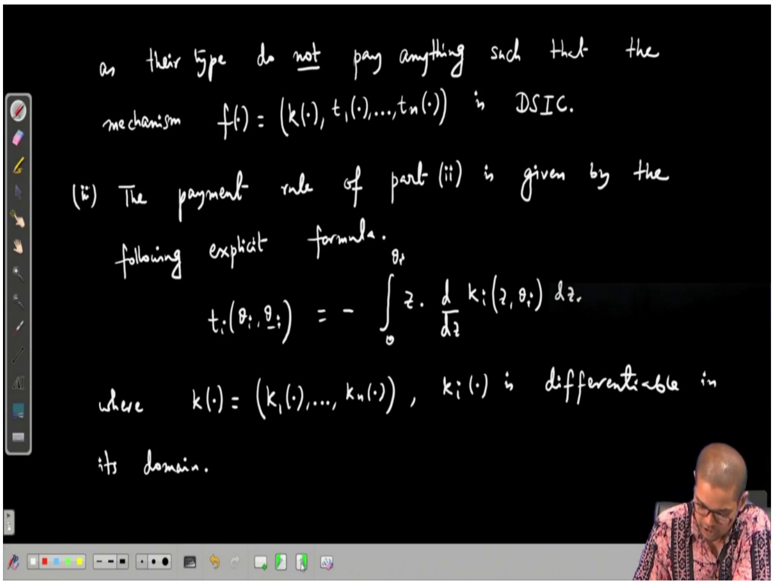

So, you cannot say that losers do not pay anything no we say that where players reporting 0 as their type do not pay anything, this condition is sort of like a normalization condition.

You see that you know if I take a dominant strategy incentive compatible social choice function and give some free say 100 rupees to all the players then losers also is not is getting some money and this 100 rupees freebies is not is not going to affect the strategic behavior because they are anyway going to get it. So, unless we assume some kind of normalizing factor on the payments payment rule cannot be unique, because if I take any payment rule and give some freebies say some thousand rupees to each players then I get another payment rule and so on.

And what this second part says is that if payment rule is essentially unique, if you do not give any freebies; that means, if players who are say in auction scenario bidding 0. That means, they will lose and they will not pay anything that is it. And the 2nd part says that if I if the allocation rule is monotone, then of course there exist a payment rule. But the payment rule is unique up to the normalization that losers do not pay anything, but here is there is no loser. So, whoever reports its type to be 0 they should they are not paying anything.

So, this such that the mechanism $f(.)=(k(.),t_1(.),\ldots,t_n(.))$ is dominant strategy incentive compatible, these are the essential 2 parts in the and there is a third part of Myerson's Lemma which gives an explicit formula of payment rule. So, which we will see now what are the 3rd part? The payment rule of part 2 is given by the following explicit formula.

What is that formula? $t_i(\theta_i,\theta_{-i})$ this is $-\int_0^{\theta_i} z\frac{d\,k_i(z,\theta_{-i})}{dz}$. So, assume the where the allocation function has this n components $(k_1(.),\ldots,k_n(.))$, think of them as like what fraction of good you know auction scenario what is the probability that player i gets that outcome that way or if the item is divisible like water or something like that liquid, which can be divided arbitrarily.

Then what fraction of good player i receives, that is like $k_i(.)$ and this and you see that we are differentiating it. So, this formula only makes sense if $k_i(.)$ is differentiable in it is domain ok. Now what is what if it is not differentiable? So, let us write another special case, so which is not differentiable and it is a step-wise function step function.

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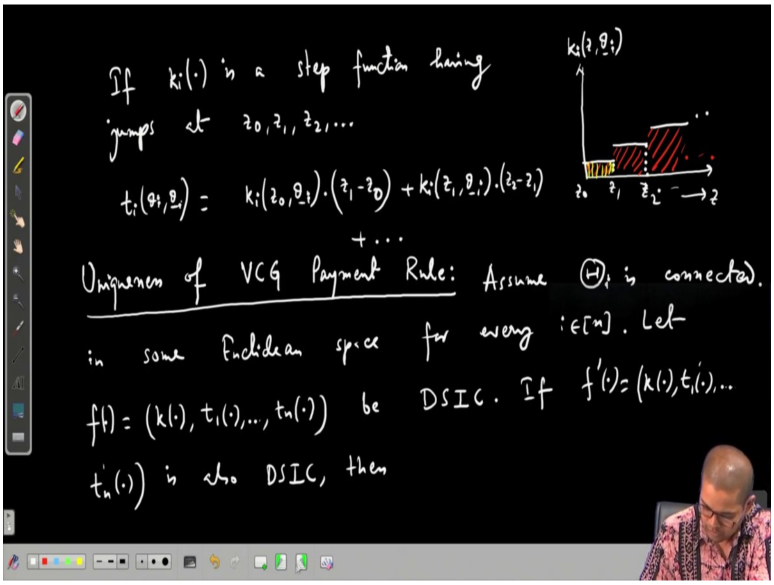

So, if k i is if it is piece-wise differentiable then also this formula is fine, you break this integration into those pieces and integrate them separately and then add that is fine. And using this formula we can see that what should be the thing for step function what should be the formula if k, k is a step function. That means, it looks like there is a z and here is $k_i(z, \theta_{-i})$, θ_{-i} is kept fixed then if it looks like this there are jumps like this this this and so on.

So, if the jumps are at say step function having jumps at say z_0. Suppose this is z_0 this is z_1 this is z_2 and so on. And then it is what it is then $t_i(\theta_i, \theta_{-i})$ is what is this area under the curve and so on. So, this is $k_i(z_0, \theta_{-i})(z_1 - z_0) + k_i(z_1, \theta_{-i})(z_2 - z_1) + \ldots$ this is the area of this yellow region and we keep adding this areas and so on.

So, this will be helpful that you know it is like it is always area under curve. Now if it is differentiable you need to you need to integrate it if it is a continuous function this allocation, if it is a discrete function like this or step wise then you need to do this you have to you need to apply this formula. But it always remains area under this curve good.

Now we finally, conclude this by stating payment functions that are essentially unique. So, uniqueness of payment rules uniqueness of VCG payment rule. So, assume θ_i is connected ok θ_i is connected in some Euclidean space for every $i \in [n]$ ok. And let a social choice function f which is allocation and payments let f be dominant strategy incentive compatible.

Then if I keep the allocation rule same and change the payment rule. Now if f' equal to which is the same allocation rule, but the payment rule is different is also dominant strategy incentive compatible.

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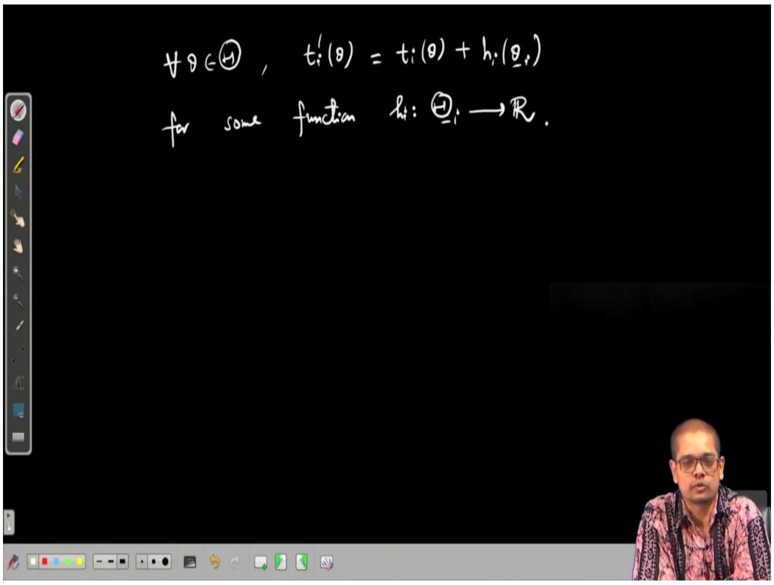

Then we have what then you know this payment must be like a shift that then for all theta for all type profile theta $t_i^{'}(\theta)$ should be equal to $t_i(\theta)$ plus maybe some normalization constant $h_i(\theta_{-i})$ ok; for some function $h_i : \Theta_{-i} \rightarrow \mathbb{R}$. So, we will stop here today.