Algorithmic Game Theory

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Lecture - 53
DSIC in Single Parameter Domain

Welcome. So, in the last class we have started studying Single Parameter Domains and we were in between proving the characterization of social choice functions in the single parameter environment which are which are implementable in dominant strategy nash equilibrium. So, let us continue that proof.

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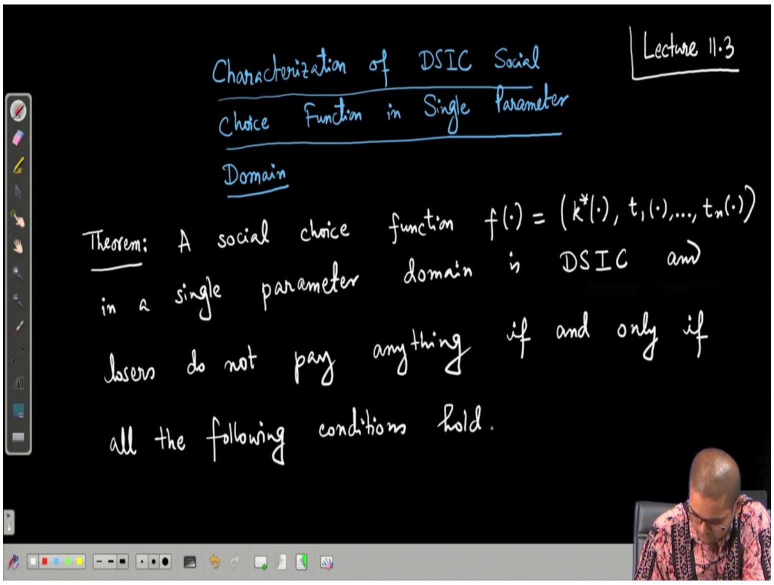

So, what was the statement let us rephrase the statement. It was characterization of DSIC social choice functions in single parameter domain. So, let us recall the statement theorem a social choice function if which has two parts – one is allocation $k^*(.)$ and payment $t_1(.),\ldots,t_n(.)$ for n players. In a single parameter domain, single parameter domain is dominant strategy incentive compatible and losers do not pay anything.

So, again observe that the term loser only makes sense in the single parameter domain because in the single parameter domain the set of all allocations scale k we have assumed that for each player i there is a subset K_i of allocations where player i wins. And, the remaining allocations; that means, $K \setminus K_i$ player i loses. So, a player i loses means that the allocation chosen by the social choice function at a particular type profile does not belong to K_i.

So, losers so, think of a social choice function where a losers do not pay anything. If for a player if it loses, then its corresponding payment should be 0. It should not get any money; it should not pay also any money. Among those social choice functions we say that our social choice function is dominant strategy incentive compatible if and only if; if and only if all the following conditions hold. What are the conditions?

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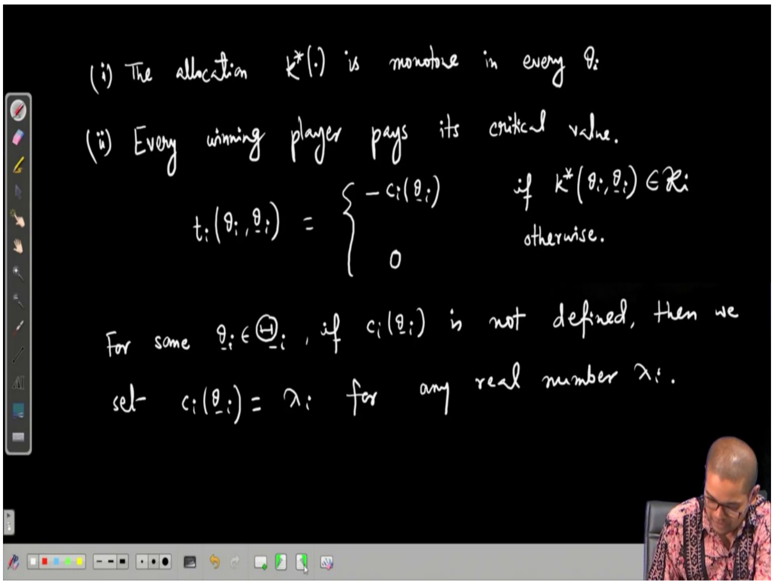

We have two conditions – one is for allocation another is for payment. The condition for allocation is the allocation function $k^*(.)$ is monotone in every θ_i. So, this is what we mean by characterization that all monotone allocation rules are implementable and they are the exactly the set of allocation rules which are implementable in dominant strategy equilibrium in single parameter domain.

Because of this theorem because this theorem says that any function which is any social choice function which is implementable must satisfy these two conditions and the reverse is also true; that means, if in if I take a social choice function in single parameter domain and if losers do not pay anything, then if it satisfies these two conditions then it is dominant strategy incentive compatible.

So, the first condition says that the allocation rule must be a monotone allocation rule in every θ_i. In each of the component it must be monotone. And every winning player p is its critical bid or critical value that is $t_i(\theta_i, \theta_{-i}) = -c_i(\theta_{-i})$ if player i wins; that means, if k star theta i comma theta minus i belongs to k i that is the meaning of winning.

And, of course, because losers do not pay anything, so, it is 0 otherwise and now, you know this critical function critical value is not defined for. So, it may not be defined for some type profile of other players, then what should we do? For some $\theta_{-i} \in \Theta_{-i}$ for some type profile of other players.

If $c_i(\theta_{-i})$ is not defined then we set $\theta_i = \lambda_i$ for any real number λ_i and because it is an if and only if statement it had two parts one is if; that means, if a social choice function satisfies these two conditions, then we need to show that it is dominant strategy incentive compatible that is the first part.

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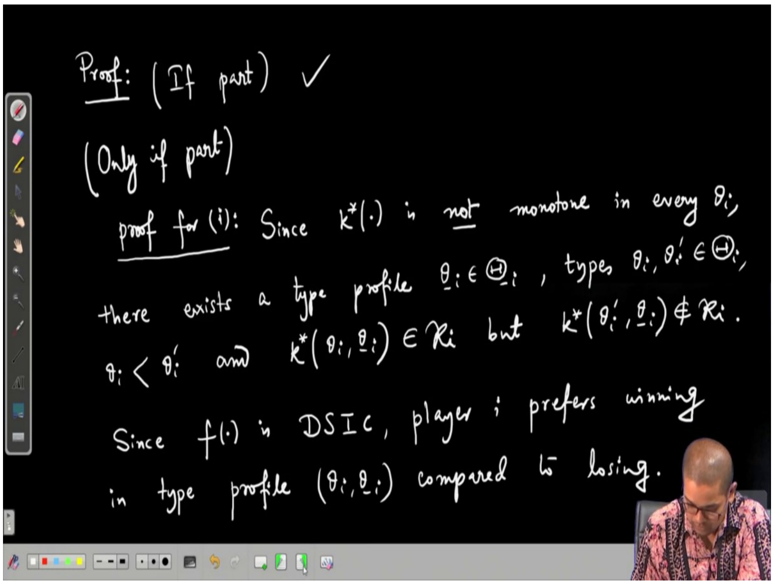

If part and this we have done in the last class, now we will prove the other only if condition; that means, if I have a social choice function which is implementable in dominant strategy equilibrium and a losers do not pay anything then it must satisfy these two conditions – conditions 1 and 2. This proof will see today only if part. So, it has two conditions. So, proof for one we first prove that the social the given social choice function if it has two parts allocation rule and payment rule.

If I take the allocation rule then it must be a monotone allocation rule that is what do you mean by proof of part 1. So, let us prove that. And, it is a proof by contradiction. So, suppose ok so, you know. So, since k star is monotone is not monotone it is a proof by contradiction. So, let. So, to begin with we will assume that $k^*(.)$ is not monotone and we will get to a contradiction. So, since $k^*(.)$ is not monotone in every θ_i, there exist there exists a type profile of other players.

$\theta_{-i} \in \Theta_{-i}$ and a type two types of player i $\theta_i, \theta_i^{'} \in \Theta_i$ with $\theta_i < \theta_i^{'}$ and if I apply the allocation rule $k^*(.)$ on (θ_i, θ_{-i}) player i wins. That means, it belongs to K_i, but if player i is type increases to some theta i prime while keeping the type profile of other players fixed, then player i all of a sudden is not winning anymore. That is what we mean by saying that the allocation rule is not monotone, ok.

Now, so, this is what this is I just rephrased what do we mean by assuming that k star is not monotone. Now, let us use what we are given. We are given that the social choice function f is dominant strategy incentive compatible. So, let us apply that. Since, f is dominant strategy incentive compatible player i prefers winning in type profile (θ_i, θ_{-i}), ok compared to losing. So, consider it a type profile (θ_i, θ_{-i}) player i has two choices. The outcome from player i perspective there are only two possible outcomes: player i can win and player i can lose.

If player i reports truthfully his type θ_i, then it will win; on the other hand, if it misreports its type to be $\theta_i^{'}$, then it will lose. So, because f is DSIC player i always prefers truth telling; that means, player i will prefer winning in type profile (θ_i, θ_{-i}) compared to losing.

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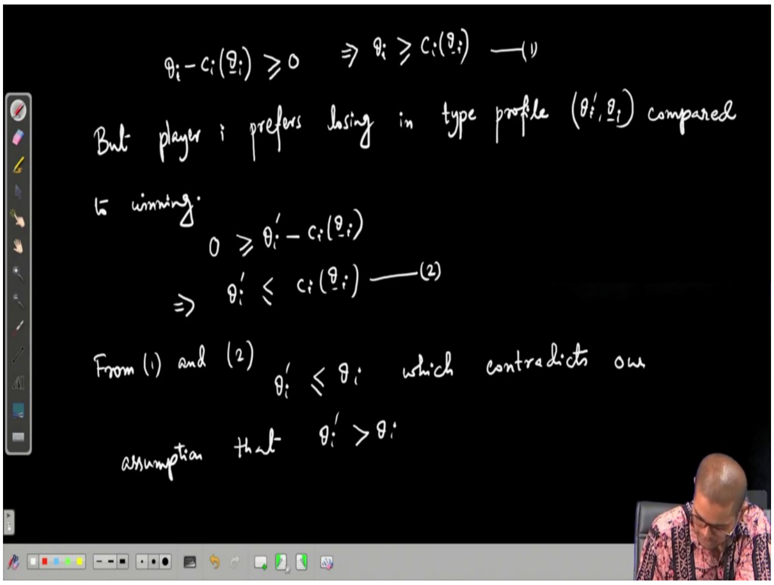

So, what is the utility of player i in the type profile (θ_i, θ_{-i}). Its valuation is the type itself, again this is from the definition of single peak domain θ_i and how much it pays it pays $-c_i(\theta_{-i})$. This is what it pays. So, valuation plus payment that is how the utility function look in the quasi-linear environment.

Now, this utility should be greater than equal to 0, which is the utility when it loses. When it loses the valuation of that allocation is 0 and it does not pay anything. So, this is this, but you know, but player i prefers losing in type profile (θ_i', θ_{-i}) compared to winning. Why? Because look at the type profile (θ_i', θ_{-i}).

Again player i would prefer truth telling because f is a dominant strategy incentive compatible mechanism. If it reports truth then player i loses, but it has option of misreporting its type and report θ_i and then it will win, but because f is dominant strategy incentive compatible player i does not want that.

So, what we have is that what is the utility of player i will why when it loses is 0 this is greater than equal to what is the utility of player i if player i misreports its type to be θ_i. Then it is then it will win, but if the valuation is θ_i'. Valuation it is its true type; even if it misreports its type to be something else the valuation is true type. The value of that item or of that allocation to player i is always its true type theta i prime, but the payment is again $c_i(\theta_{-i})$.

So, what do we get here from here? That $\theta_i^{'} \leq c_i(\theta_{-i})$ and from the above we get that $\theta_i \geq c_i(\theta_{-i})$. So, let us call this one inequality 1 and this inequality 2. So, from 1 and 2, what do we have is that $\theta_i^{'} \leq \theta_i$ which contradicts our assumption that $\theta_i^{'}$ is greater than θ_i this is what we assumed in the beginning. Here $\theta_i^{'}$ is greater than θ_i. This contributes our assumption. Hence the allocation rule $k^*(..)$ must be monotone which proves part 1.

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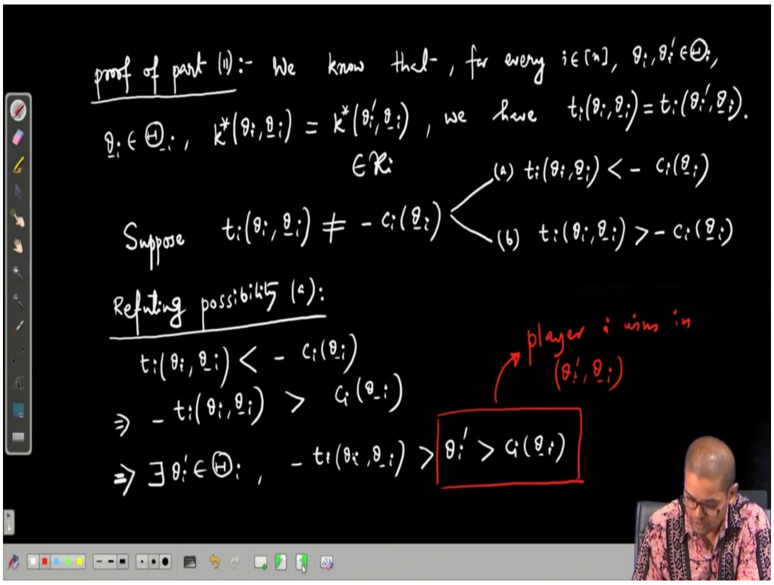

Now, proof of part II. So, we already know that the allocation rule is monotone and we know that the payment depends on the type profile of other players. So, that we have proved before. So, write we know that for every $i \in [n]$, $\theta_i, \theta_i^{'} \in \Theta_i$ types $\theta_{-i} \in \Theta_{-i}$; we are taking two type profiles (θ_i, θ_{-i}) and $(\theta_i^{'}, \theta_{-i})$. That means, only the type of player i changes everything else remains same and the allocation is same.

And, $k^*(\theta_i, \theta_{-i})$ is same as $k^*(\theta_i^{'}, \theta_{-i})$ for any such things we have recall allocation sorry payment depends on payment of player i depends on its type θ_i only through allocation. So, if allocation does not change payment also does not change. If the type profile of other players remains same; here type profile of other players has remained same θ_{-i} the type of player i has changed but the allocation has not changed.

So, in both the type profile, the payment to player i must be the same. These we have proved couple of lectures ago. $t_i(\theta_i, \theta_{-i}) = t_i(\theta_i^{'}, \theta_{-i})$ payment to player i remain same because allocation has remain same and the type profile of other players has also remained same. Now, suppose what we need to show? We need to show that the payment is equal to and suppose this is belongs to K_i because losers do not pay anything that is part of the assumption.

So, if player i loses then we do not have to show anything because it follows from the assumption, only if it wins then we need to show that the payment should be it is critical value. So, again it is a proof by contradiction. Suppose not suppose the payment in this particular type profile $t_i(\theta_i, \theta_{-i})$ here in this type profile player i wins, but this is not equal to its critical value $c_i(\theta_{-i})$ ok. So, if it is not equal to this then only two cases can happen case one is only possibility is let us call it $t_i(\theta_i, \theta_{-i})$.

This is less than $-c_i(\theta_{-i})$ or it is more than $t_i(\theta_i, \theta_{-i})$ this is more than $-c_i(\theta_{-i})$. So, we will refute both the possibilities thereby proving that $t_i(\theta_i, \theta_{-i})$ must be equal to $-c_i(\theta_{-i})$. So, refuting possibility a let us repeat that. So, what do we have that we have $t_i(\theta_i, \theta_{-i})$; this is less than $-c_i(\theta_{-i})$ which implies that $-t_i(\theta_i, \theta_{-i})$ this is greater than $c_i(\theta_{-i})$.

Now, because this is strictly greater; that means, and because it is an interval because that each type is a real interval there exists a type $\theta_i^{'} \in \Theta_i$ such that which is in between these two values $-t_i(\theta_i, \theta_{-i})$, this is greater than $\theta_i^{'}$ is greater than $c_i(\theta_{-i})$. And, you see that what is critical value? Any bid it is any for any type which is strictly more than critical value player i wins.

So, from the second part that this part we get that player i wins in $(\theta_i^{'}, \theta_{-i})$. This is from the definition of critical value and using the first part that the allocation rule is monotone. Any type which is strictly more than critical value player i will win because it is from the definition of critical value and the monotonicity of allocation rule.

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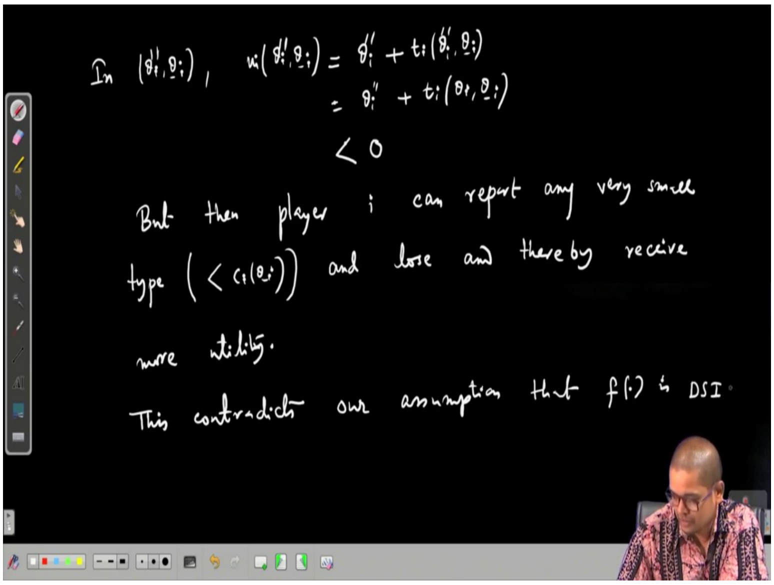

So, player i wins. Then what is its utility? In the type profile in $(\theta_i^{'},\theta_{-i})$ it is type the utility $u_i(\theta_i^{'},\theta_{-i})$ because it will reveal truth because if is a dominant strategy incentive compatible mechanism. This is $\theta_i^{'}+t_i(\theta_i^{'},\theta_{-i})$, but you see that in (θ_i,θ_{-i}) let us call it double prime because $\theta_i^{'}$ we have already used double prime.

Now, because player i is winning in (θ_i,θ_{-i}) in type profile and so, this is utilities this and in $(\theta_i^{''},\theta_{-i})$ also player i wins and type profile of other players has not changed, allocation has not changed, player i still wins. So, this is $\theta_i^{''}+t_i(\theta_i,\theta_{-i})$.

But, $t_i(\theta_i,\theta_{-i})+\theta_i^{''}$ from this part shift $-t_i(\theta_i,\theta_{-i})$ to the other side right hand side we get $t_i(\theta_i,\theta_{-i})+\theta_i^{''}<0$. So, player i by winning its utilities strictly less than 0, but then player i can report any very small type in particular anything which is less than its critical bid $c_i(\theta_{-i})$ can report any very small type and lose.

And, thereby receive more utility because if it loses then it is value it is utility is 0, but in this right profile $(\theta_i^{''},\theta_{-i})$ by telling truth it get it gets negative utility. So, it is so, in this particular type profile $(\theta_i^{''},\theta_{-i})$ player i is better of losing which contradicts our assumption that f is DSIC this contradicts our assumption that f is DSIC.

So, which refutes that this first part cannot be the case a part. Similarly, I would encourage you to mimic the same technique and prove that the second part is also not possible and once you refute both the parts the only option left is that payment must be equal to critical value which will conclude the proof. So, refuting part b I am leaving it as a homework. So, we will stop it here today.