Algorithmic Game Theory
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 52
Single Parameter Domain

Welcome. In the last class we have defined Single Parameter Domain and we will continue single parameter domain in this class. We will study what are the social choice functions, which are implementable in single parameter domain.

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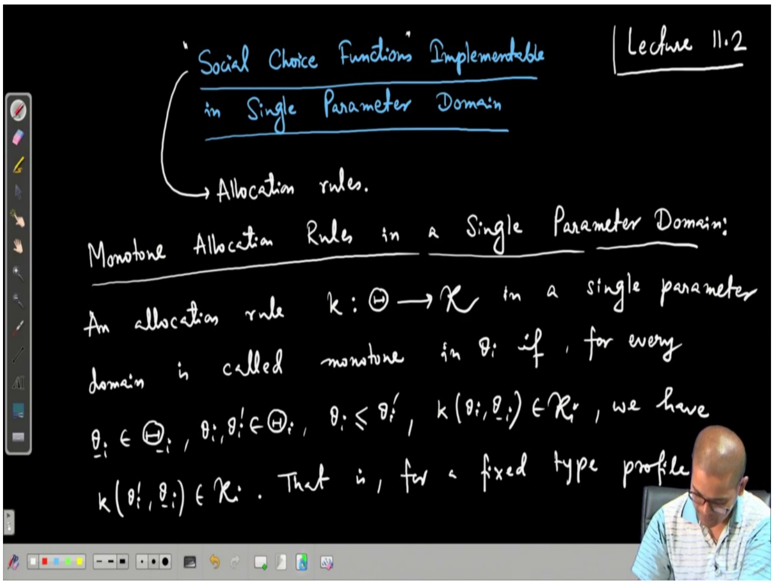

So, social choice functions implementable in single parameter domain. So, towards that let us define, what is called monotone allocation functions. Again by social choice functions, I mean social choice function has two part one is allocation rule another is payment rule.

And in quasi linear setting when we mean that, when we ask what are the social choice functions implementable; we it should be it is it should be understood that we mean that what are the allocation rules for which there exists a corresponding payment rule. So, that the allocation rule plus payment rule thereby giving the social choice function is implementable in dominant strategy Nash equilibrium, domain strategy equilibrium.

So, by social choice function, we basically mean what are the allocation rules. So, the answer is monotone allocation rules monotone allocation rules. Of course, this makes sense only in the single parameter domain in a single parameter domain. So, what are they? So, an allocation rule $k:\Theta \rightarrow K$ is the set of all allocations in a single parameter domain is called monotone in θ_i if for every $\theta_{-i} \in \Theta_{-i}$.

For every type profile of other players, if we if I have to if I take two types $\theta_i, \theta'_i \in \Theta_i$ with say because, recall that θ_i and θ'_i these are real numbers. In single parameter domain capital theta is an interval and they are real it is a real interval. And actually that is why the name single parameter domain. The domain theta i is a single parameter it is a single real number.

Suppose, $\theta_i \leq \theta'_i$ and we have that $k(\theta_i, \theta_{-i}) \in K_i$. That means, when the type profile of other players is θ_{-i} with θ_i as type of player i, player i wins. Then for every this, we have $k(\theta'_i, \theta_{-i}) \in K_i$. So, what does it say?

It says that fixed type profile of other players θ_{-i} and if at a type θ_i player i wins and then player i increases its type to θ'_i, then player i should continue to win. That is what is meant by monotone allocation rules. One player i wins, then if other players type profiles are fixed, then increasing type of player i continues to win.

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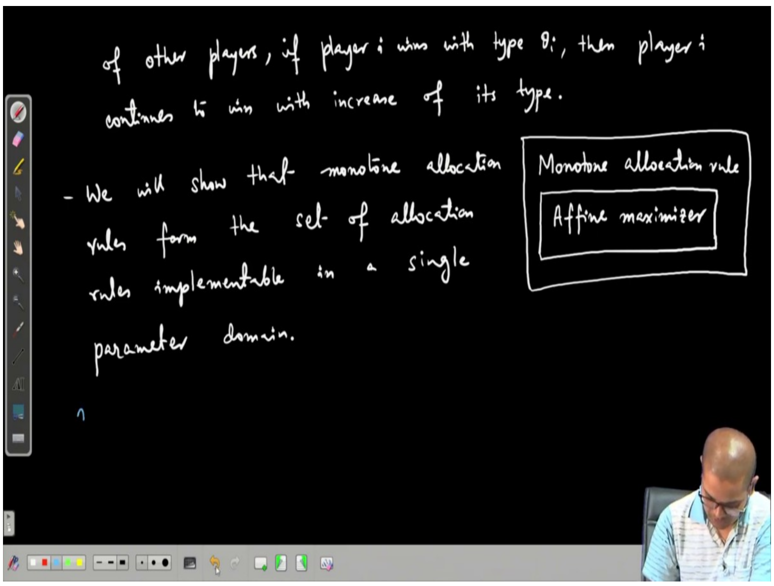

So, that is for a fixed type profile theta minus i of other players if player i wins with type θ, then player i continues to win then player i continues to win with increase of is with increase of its type ok. And the main result of single parameter domain is that all monotone allocation rules are implementable in single parameter domain.

Now, because single parameter domain is a special case of is a special case of is a more restricted setting than quasilinear environment, anything that is implementable in quasi linear environment that will continue to get continue to be implementable in single parameter domain. And with this understanding, what we get is that we know what are the implementable social choice functions, the affine maximizers.

These are the implementable allocation rules in quasi linear environment affine maximizers and we are claiming that we will show that monotone allocation rules are implementable in single parameter domain. So, the monotone allocation rules form a super set of affine maximizer or saying in the saying the same thing in different word, each affine maximizer is a monotone allocation rule.

So, it is monotone allocation rule. So, monotone allocation rule forms a strict super subset super set of affine maximizers ok. So, here is the main claim, which we will show that we will show that monotone allocation rules form the set of allocation rules implementable in a single parameter domain. So, towards that we need to define a notion called critical value function. So, let us define it. Critical value function, let us define the next page.

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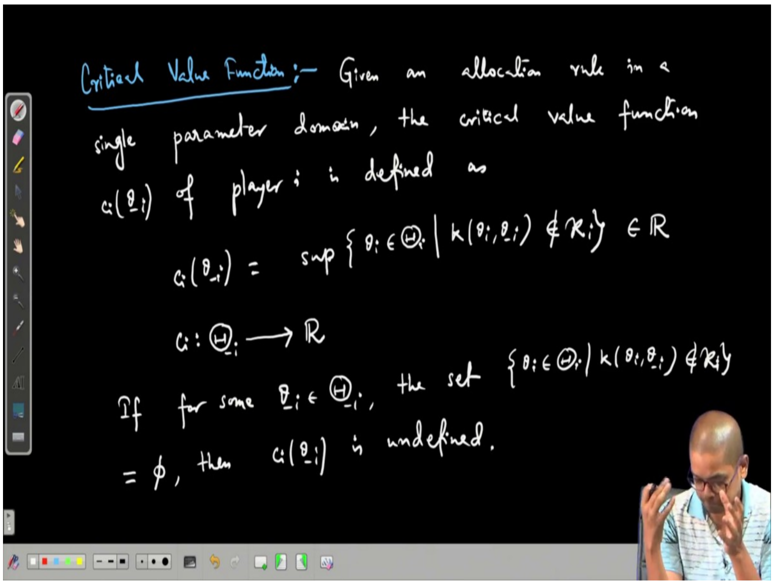

Critical value function: So, given an allocation rule in a single parameter domain, the critical value function $c_i(\theta_{-i})$. For each player we have a critical value function and it is a function of type function from type profile set of all type profiles of other players. The critical value function $c_i(\theta_{-i})$ of player i at the critical value function theta i is defined as $c_i(\theta_{-i})$ is supremum of this set; $\theta_i \in \Theta_i$, such that $k(\theta_i, \theta_{-i}) \notin K_i$.

Recall, in a single parameter domain for each player, there is a subset of allocations K_i, where in some sense player i wins. And now, what is the critical value function? So, the value of the critical value function at a particular type profile theta minus i is defined to be supremum of all θ_i. Θ is these are this real number. Supremum of all types, where player i does not win. So, this is $c_i(\theta_{-i})$. And of course, so, this critical value function c_i is a function from type profile of other players to Θ_i ok.

And θ_i is real numbers. If for some $\theta_{-i} \in \Theta_{-i}$, the set if the set is empty set, if the set Θ_i, $k(\theta_i, \theta_{-i}) \notin K_i$. If the set is empty set then simply then c i of theta minus i is undefined. And we do not put any special value ok.

So, next we show that this critical value functions using critical value functions, we will show that this monotone allocation rules are exactly the set of all allocation rules implementable in single parameter environment.

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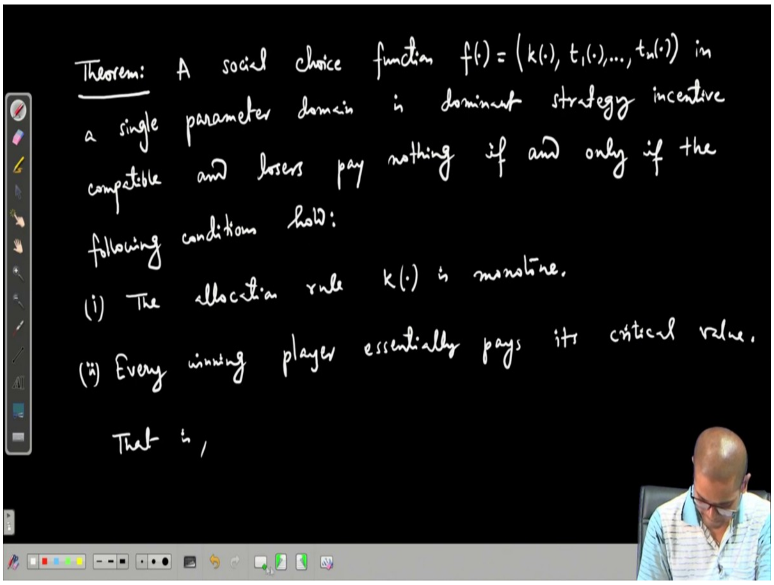

So, theorem: A social choice function f which has two parts allocation and payment, in a single parameter domain is dominant strategy incentive compatible and losers pay nothing; that means, if for some player it is the allocation is allocation does not belong to K_i, for some player i allocation does not belong to K_i, then the its payment t_i will be 0. It do not pay anything, it do not get anything. If this hold, this will hold if and only if and only if the following conditions hold.

1st, the allocation rule is monotone. Allocation rule k is monotone. And payment rule, it is like every player essentially pays a critical bid critical value. So, every winning player, every winning player essentially pays its critical value.

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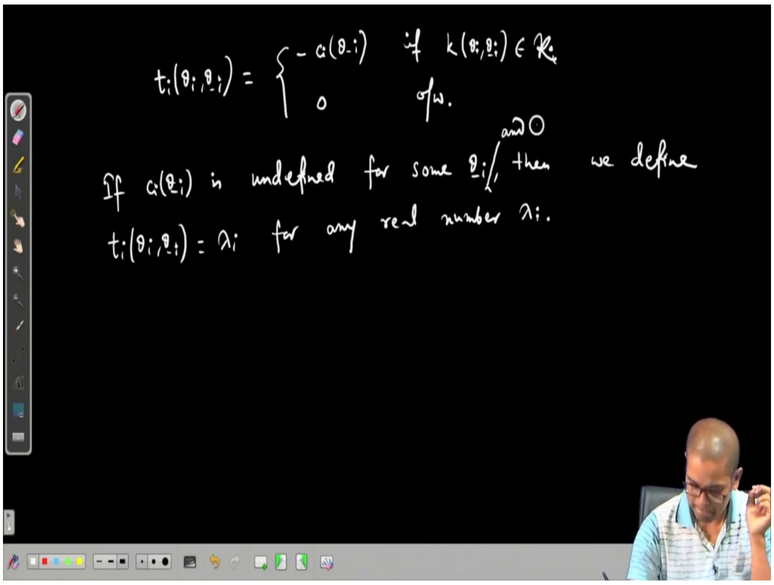

That is $t_i(\theta_i, \theta_{-i}) = -c_i(\theta_{-i})$ if player i wins $k(\theta_i, \theta_{-i}) \in K_i$ and 0 otherwise; because losers would not pay anything. For now you know it can happen that for some θ_{-i}, $c_i(\theta_{-i})$ is undefined; then what should we do?

Then right if $c_i(\theta_{-i})$ is undefined for some θ_{-i}, then we define $t_i(\theta_i, \theta_{-i}) = \lambda_i$. So, for a player pick a real number λ_i and if $c_i(\theta_{-i})$ is not defined, then just to just define the payment to be something fixed equal to λ_i ok. Is not defined for some θ_i and player i wins. And or actually, let us we define let us define $c_i(\theta_{-i})$. So, if it is needed, it will be used.

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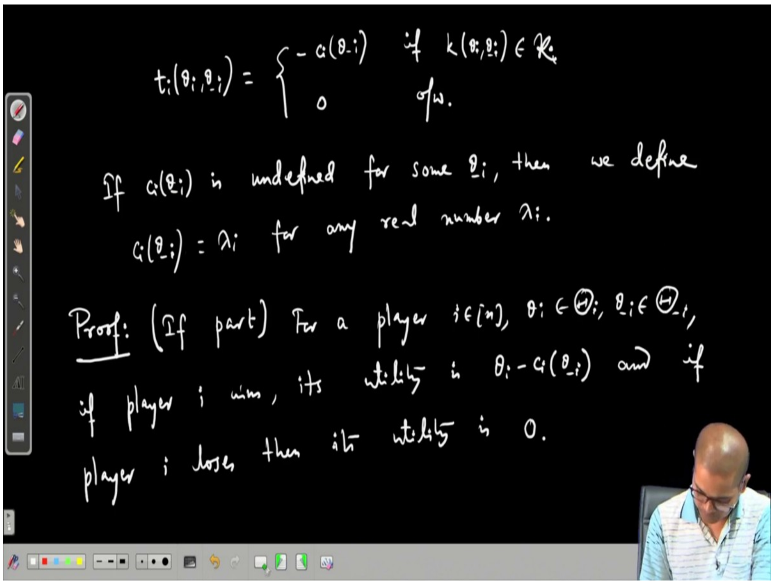

So, $c_i(\theta_{-i})$. So, if player i wins, then it will pay λ_i and if it loses it will be 0 ok. Proof: First if part, because it is a if and only statement we need to show both directions; first if part ok. So, let us see. So, for a player $i \in [n]$ and say $\theta_i \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$, if player i wins, its utility is again valuation plus payment.

What is valuation? Valuation is θ_i that is why that is what is that is what is assumed in this definition of single parameter domain. So, valuation is θ_i and payment is $-c_i(\theta_{-i})$. And play if player i wins, then it situates 0 and if player i loses then its utility is 0, because if it loses the its valuation is 0 and it does not pay anything. Now, why this is dominant strategy incentive compatible?

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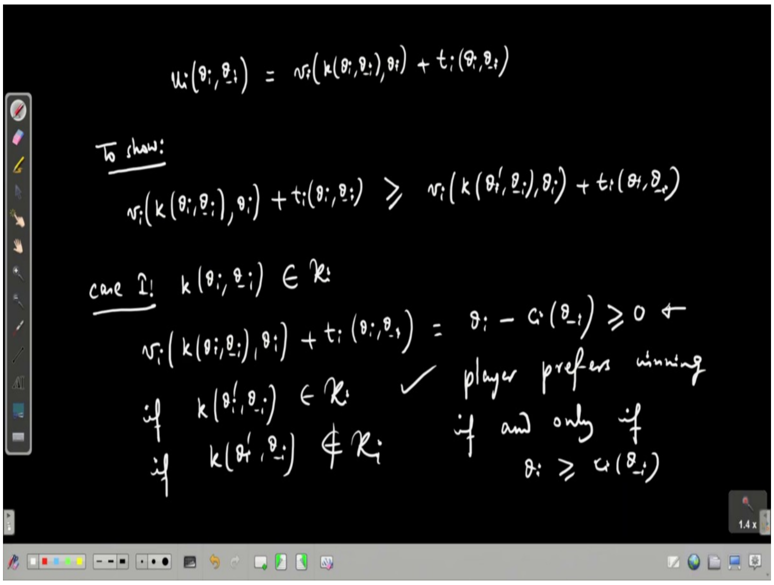

So, what is $u_i(\theta_i, \theta_{-i})$? It is as i said this is $v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})$. Now, if player i misreports, then we need to show that.

So, to show that this particular utility $u_i(\theta_i, \theta_{-i})$ which is $v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})$ this is greater than equal to if player i's true type is θ_i, but still it reports $\theta_i^{'}$, then what will be its utility?

Its utility will be $v_i(k(\theta_i^{'}, \theta_{-i}), \theta_i)$. This will be its valuation, because its true type remains θ_i. Although, it has reported $\theta_i^{'}$ plus $t_i(\theta_i, \theta_{-i})$. Now, 2 cases, let us do 2 cases. Case I, that player i wins in (θ_i, θ_{-i}) , $k(\theta_i, \theta_{-i}) \in K_i$, then what is its utility?

$v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})$ its valuation is theta i because it wins. So, θ_i and payment is critical value $c_i(\theta_{-i})$. Now, 2 cases can be can happen again. If $k(\theta_i^{'}, \theta_{-i}) \in K_i$, that means, player i continues to win in $(\theta_i^{'}, \theta_{-i})$; then its utility remains same. Its utility remains $\theta_i - c_i(\theta_{-i})$. So, it does not gain by lying.

On the other hand, so, in this case this is fine. Now, on the other hand if $k(\theta_i^{'}, \theta_{-i}) \notin K_i$ then player i loses and in this case its utility is 0. Now, you see that this is always greater than equal to 0. So, player i will prefer player i will.

So, what do we get? So, player i. So, this is this holds if this is greater than equal to 0. So, player i prefers winning; if and only if this inequality holds. That means, $\theta_i - c_i(\theta_{-i}) \geq 0$; that means, $\theta_i \geq c_i(\theta_{-i})$.

This is exactly the definition of critical bit. Player i will prefer winning, if it is true type theta i exceeds its critical bit; otherwise it will prefer losing. So, similarly you can do case II, if $k(\theta_i, \theta_{-i}) \notin K_i$, player i loses in the type profile (θ_i, θ_{-i}), the same thing will continue. So, in the next class we will continue this proof and we will show the other direction the only direction; which is which is more non trivial ok. So, we will stop here.