Algorithmic Game Theory
Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 05
Nash Equilibrium

Welcome. So, in last few lectures we were studying various equilibrium concepts we started with strongly dominant strategy equilibrium, then we weaken it and what we get is called sorry weakly dominant strategy equilibrium. And then we weakened that further and we got very weakly dominant strategy equilibrium. Then we proved in the last class, our first theorem a non-trivial important theorem is that the for the second price auction bidding valuations is a weakly dominant strategy equilibrium.

So, in some sense this is general weakly dominant strategy equilibrium is a weaker weakening of strongly dominant strategy equilibrium, but still if you go through the examples of or games of or normal form games like, Battle of Sexes coordination game and so on, they do not even have weakly dominant strategy equilibrium. So, that that is when the genius of John Nash comes into picture and what he defines is called Nash equilibrium ok.

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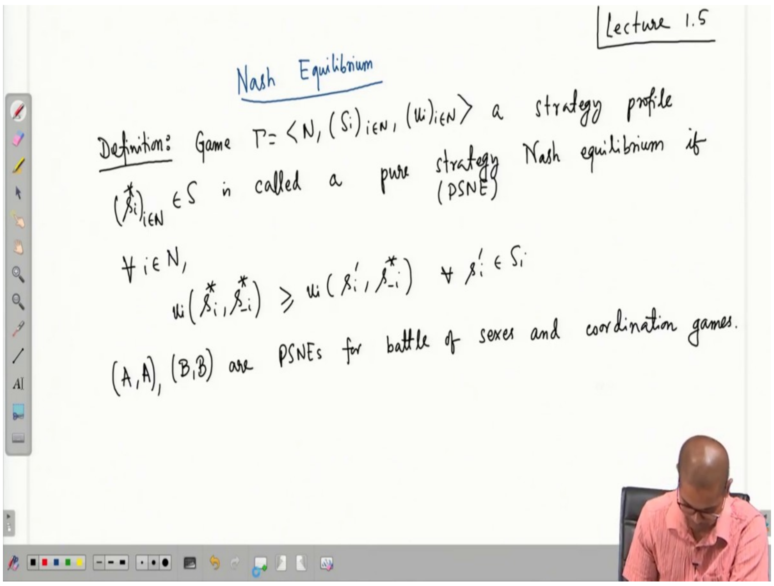

So, what is Nash equilibrium? So, let us first define it, given a game, gamma equal to N set of players, strategy set of all the players, utilities of all the players. A strategy profile

$(s_i^*)_{i \in N} \in S$ where $s_i \in S_i$ is called a is called a pure strategy Nash equilibrium, if unilateral deviation is not beneficial for all the players. If for all players $i \in N$, this condition holds. Utility of player i when it is playing s_i and other players are playing s_{-i}, this is greater than equal to $u_i(s_i^{'}, s_{-i})$, this should happen for all $s_i^{'} \in S_i$.

So, given so let us denote the term s_i^*, we typically denote star to denote equilibrium. So, a strategy profile s_i^* is called a pure strategy Nash equilibrium, if given no other players are deviating or continuing to play s_i s, play according to this strategy profile. The unilateral division if player one only deviates from the strategy profile and plays some other strategy $s_i^{'}$, then it will not gain by doing so. So, this is called a pure strategy Nash equilibrium.

So, let you check that for coordination games players coordinating (A, A) and (B, B), both are pure strategy Nash equilibriums, this is abbreviated as PSNE, Pure Strategy Nash Equilibrium. (A, A) and (B, B) are PNSEs for battle of sexes and coordination games, ok. But, you know you still see that ok; so we have weakened the notion of very weakly dominant strategy equilibrium, what if by weakened.

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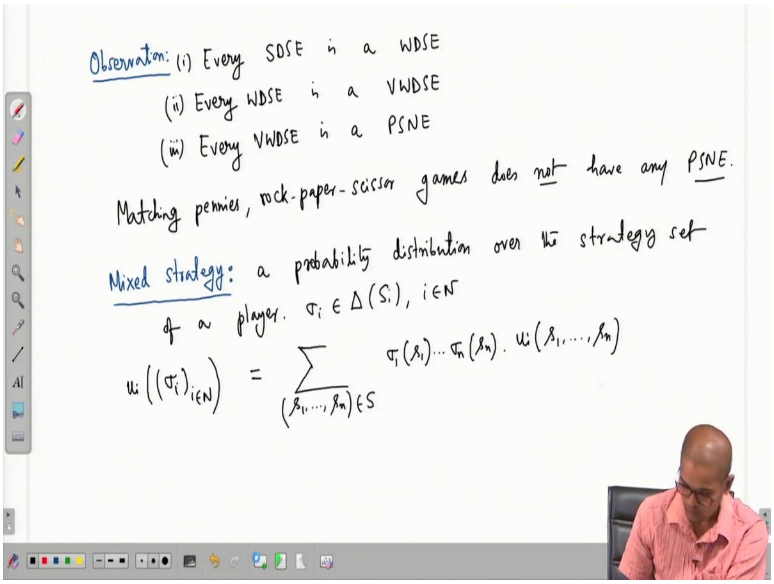

So, let me formally write what do you mean by weakening. So, write it as a observation and I will let the proof to you. Observation, every strongly dominant strategy equilibria is a weakly dominant strategy equilibrium. In this sense, weakly dominant strategy equilibrium is a weakening of the notion of strongly dominant strategy equilibrium. Every weakly dominant strategy equilibria is a very weakly dominant strategy equilibrium.

In that in this sense these further weakening and similarly, you can show that every very weakly dominant strategy equilibrium is a pure strategy Nash equilibrium. So, yes, so whenever we weaken the solution concept the equilibrium concept, then more games have that equilibrium. But still, there are games which does not have PSNE, for example, you know matching pennies and then say rock-paper-scissor; rock-paper-scissor games does not have any PSNE and this is where the genius of john Nash comes into picture.

So, instead of weakening the notion further john Nash asks tells ok, why players has to play only deterministic strategies, they can or they may be allowed to play randomized strategies. They may be allowed to uniformly or they may be allowed to draw a draw their strategy at front time, at play time from a distribution. So, and that is the key concept.

So, define what is called mixed strategy, what is mixed strategy it is nothing but a probability distribution over the strategy set of a player. So, the mixed strategy of player i is a probability distribution over its strategy set S_i. So, in that sense the normal the other strategies are called pure strategy and that is why it is called pure strategy Nash equilibrium.

But how do we define utilities in when players allowed to play mixed strategies. So, in that case in, in that scenario we replace mixed we replace, utility with expected utility, because probabilities are involved. So, what is the utility? So, suppose $(\sigma_i)_{i \in N}$ is a mixed strategy profile; that means, what is σ_i? σ_i is a probability distributions over $S_i, i \in N$.

So, $\Delta(S_i)$ is the set of all probability distributions over S_i. So, what is the utility of player i say in this strategy profile, in this mixed strategy profile? Very simple, you go over all strategies, all strategy profiles. What is the probability that this particular strategy is played? This is the probability that player 1 plays s_1, this happens with probability $\sigma_1(s_1)$, what is the probability that and so on.

Player n plays s_n is $\sigma_n(s_n)$ this is the probability; that means, $\sigma_1(s_1) \times \ldots \times \sigma_n(s_n)$ with this probability, this particular strategy profile is played. And what is its utility of player i in this strategy profile? $u_i(s_1, \ldots, s_n)$. So, this entire sum is nothing but the expected utility of player i when players play according to this strategy profile, mixed strategy profile.

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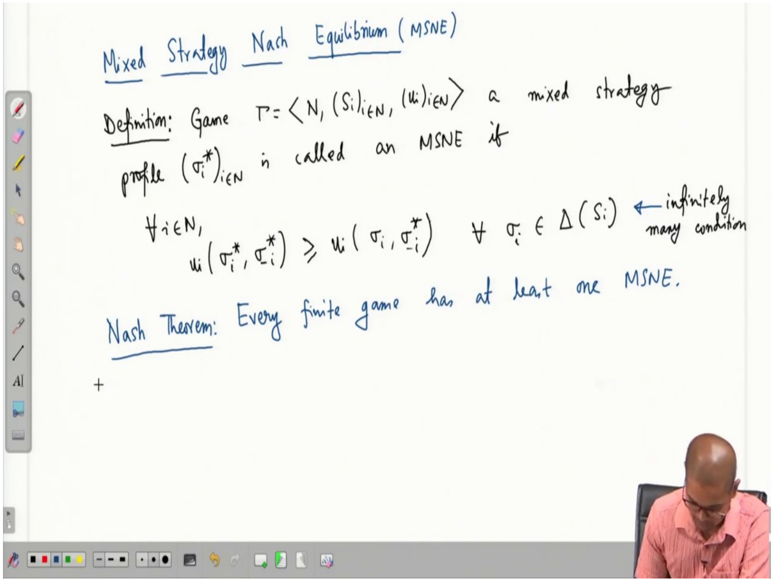

So, from here we define what we called a mixed strategy Nash equilibrium, mixed strategy Nash equilibrium. So, what is mixed strategy Nash equilibrium? Again, definition: given a game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ given a game in normal form a mixed strategy profile, mixed strategy profile, sigma i star typically we use star in equilibrium, i in N is called a an MSNE mixed strategy Nash equilibrium.

If again what is the definition? Unilateral deviation should not be beneficial for any player, if for all player i in N, utility of player i when it plays σ_i^* given other players are also playing σ_{-i}^* according to this strategy profile. This is greater than equal to $u_i(\sigma_i^*, \sigma_{-i}^*)$ this should happen for all mixed strategy $\sigma_i \in \Delta(S_i)$ ok.

So, this is called a mixed strategy Nash equilibrium and the most striking result is that every finite game has mixed strategy Nash equilibrium. And that is the famous Nash theorem which we will we will prove we will not prove, but let us state. So, what is Nash theorem? Every finite game has at least 1 MSNE, this is called the universal property of MSNE, this MSNE is present in every finite game.

What is the finite game? That means, the number of players has must be finite, the strategy set must be finite. So, for example, the auction example is not a finite game, although the number of players is finite the strategy sets are infinite, but all other games like battle of sexes, coordination games and say tragedy of common all are finite games and the celebrated Nash theorem states says that every finite game must have a Nash equilibrium ok.

Very good, but you know it sometimes it looks the condition for mixed strategy Nash equilibrium may sometime look a bit scary. For example, the condition for pure strategy Nash equilibrium or weakly dominant strategy equilibrium and so on, it is like there are finitely many conditions.

Something should hold for all strategy profiles for all strategies, but here you see that here we have infinitely many conditions infinitely many conditions, this inequality should hold for all σ_i in for all probability distribution σ_i and even if the strategy set is finite the number of probability distributions is infinite. So, that is where the famous indifference principle comes into picture, here is a let me write in a separate picture.

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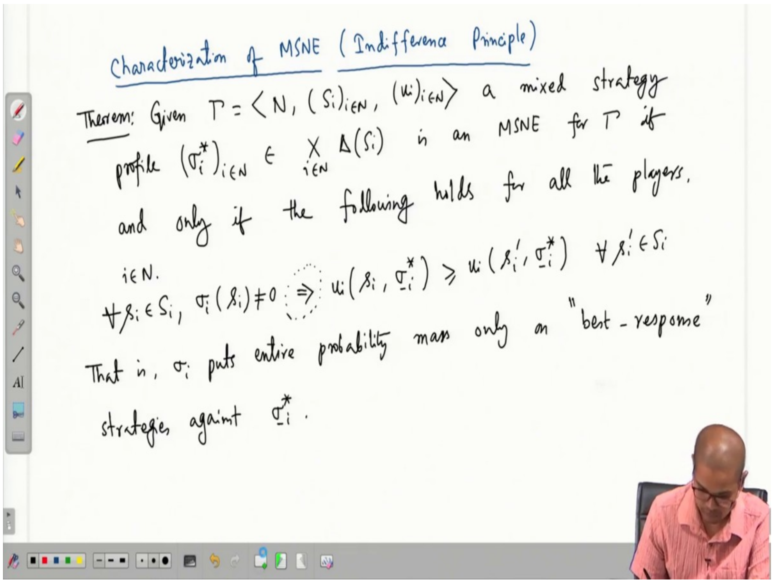

So, characterization of MSNE is called indifference principle, another very useful theorem let me write. So, given a game in normal form $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$, a mixed strategy profile; a mixed strategy profile $(\sigma_i^*)_{i\in N} \in \prod_{i\in N} \Delta(S_i)$ set of all probability distributions is an MSNE for gamma if and only if it is a characterization if and only if, we will write a finitely many condition.

And this theorem says that if that finitely many conditions are satisfied, then the strategy profile is a mixed strategy Nash equilibrium. So, if and only if the following holds for all the players, all the players $i \in N$. So, what should hold that say for all the players say i in N. So, for all strategy set s_i, for all $s_i \in S_i$ such that $\sigma_i(s_i)$ is not equal to 0.

So, look at the probability distribution the mixed strategy σ_i and consider those strategies $s_i \in S_i$, where σ_i puts non-zero probability mass, those should satisfy the following condition. So, for every strategy S_i, if this is the condition. So, if it gets non-zero probability mass, then it is it should be one of the best strategies to play for player i when other players are playing σ^*_{-i}.

This is greater than equal to $u_i(s_i', \sigma^*_{-i})$, for all $s_i' \in S i$. This is sometimes called that is we write, we introduce a useful terminology this is called best response, that is or let me write here, that is σ_i puts entire probability mass only on best response strategies against σ^*_{-i} ok.

So, it all the probability mass is put only on the best response strategies. In particular we can say that if some particular strategy is getting 0 probability, then it may be best response. So, it may be the case that it is best response. So, this is one way implication, this is very important, it can leave out it can. So, for a strategy profile σ^*_{-i} there could be more than one strategy which maximize utility of player i.

And among those many best strategies best responses the sigma i may leave out such some options some strategies, but you know, but still all the probability must should fall only on best responses. It should not give any probability to any strategy which is not best response, that is what it says. So, this proof is not difficult, but we will do it in the next class and we finish today with some examples. So, new page.

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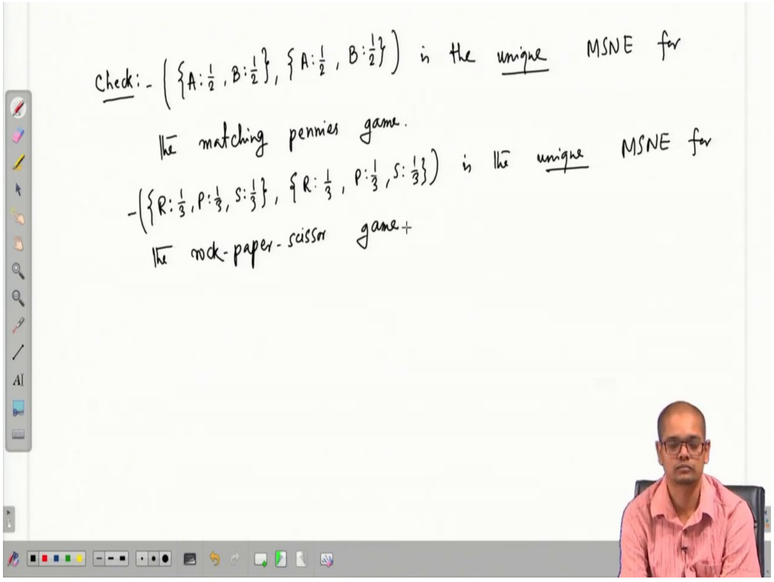

So, I let you check, first check is that if you look at the remember the matching pennies game. So, playing A with probability half and B with probability half, this is for player 1. And for player 2 also playing A with probability half and playing B with probability half, this is a this is not a, the unique mixed strategy Nash equilibrium for the matching pennies game.

Then you can, I let you check that you know for the rock-paper-scissor game playing rock with probability one-third, paper with probability one-third and scissor with probability one-third for player 1. And same for player 2, rock with probability one-third, paper with probability one-third and scissor with probability one-third is the unique mixed strategy Nash equilibrium for the rock-paper-scissor game ok.

Thank you.