Algorithmic Game Theory
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Lecture - 49
Weighted VCG

Welcome. In the last class we have seen various examples of VCG mechanisms; we will continue those seeing more examples in today's class. So, let us begin.

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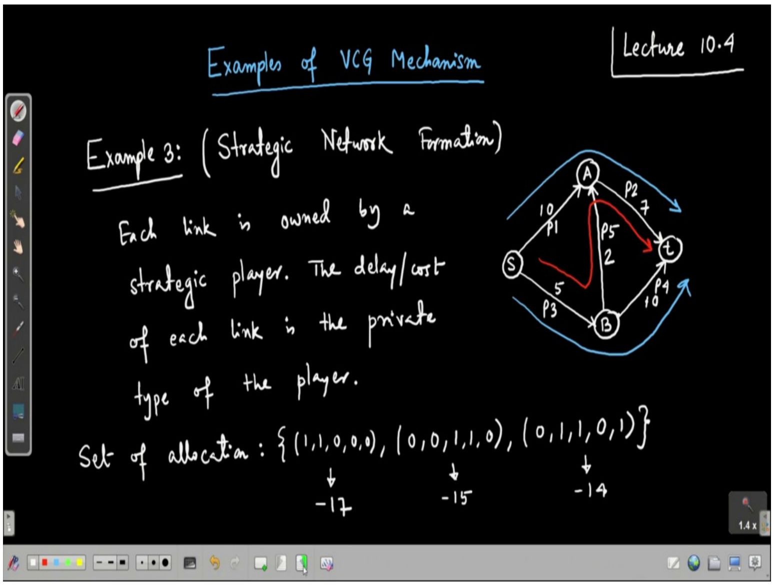

So, examples of VCG mechanism: In the last class we have seen two examples selling more items one seller selling multiple identical items that was our first example and the second example was of combinatorial auction one seller is selling two different items A and B to 3 buyers.

Now, we begin with our third example is not of auction, but something different let us see which is strategic network formation strategic network formation. So, suppose this is I have a network these are the nodes s t A B each link is held by a strategic player is owned by a strategic player and each link has a certain delay which is the private information of that player. The delay or cost whatever terminology we can use the delay or cost of each link is the private type of the player.

But here comes the beauty of VCG mechanism because we will ensure that the allocation rule that we use is a allocatively efficient allocation rule and the payment rule will be a Groves payment rule in particular it will be a Clarkes payment rule the resulting mechanism will be dominant strategy incentive compatible due to Groves theorem and hence we can assume without loss of generality that all players report their true time in the best of their own interest.

So, that makes the job of mechanism design much more easy we need just need to ensure that the allocation rule is allocatively efficient and payment is according to Clarkes payment rule or Groves payment rule. So, let us write what are the delays may be. So, this is say delay 10, this is sub 7, this is 2, 5 is 10 although these are private each player voluntarily report these valuations ok.

So, what are the set of allocations? The mechanism designer is looking for a s to t path. So, each allocation must be a path of path from s to t. So, suppose this is player 1, this is player 2, this is player 3, this is player 4 each edge is a player this is player 5. So, one path is say a to A to t which is equivalent to choosing player 1 and player 2. Another path is a to B to t which is equivalent to choosing not 1 not 2 choose 3 choose 4 and choose 5 not choose 5.

Another path is S to B to A to t. So, do not choose 1 sorry do not choose 1; that means, 0 here, then choose link 2 1 choose link 3 do not choose link 4 choose link 5. So, these are the three paths and we have these three allocation rule. Now which allocation rule is allocatively efficient? Allocative efficiency mean means is in this case minimizing the total cost.

So, the total cost of this particular allocation is 10 plus 7, 17. So, let us write minus because it is cost minus 17, the cost of sum of cost of this particular allocation is to B to t is 5 plus 10 15. So, let us write minus 15 and s to B to A to t is the cost of s to B is 5 plus 2 plus 7, 14 minus 14.

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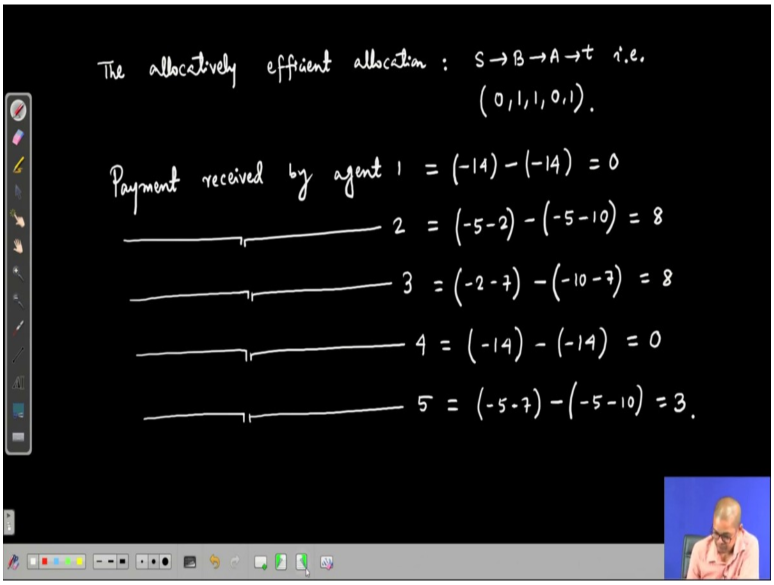

So, the allocatively efficient allocation is s to B to A to t. So, the allocatively efficient allocation is s to B to A to t which is same as that is same as picking the allocation vector 0, 1 comma 1 comma 0 comma 1 ok. So, this is the allocation that any allocatively efficient allocation rule will pick. What are the payments? Payment received by. So, we have observed that whoever is not getting anything or whoever values this allocation to be 0 namely player 1 and player 4 they do not pay anything.

So, let us see again the payment received by agent 1 is sum of valuations of this allocation is of all the players except player 1 is minus 14 as we have seen minus in the absence of player 1 the same allocation continues to be the allocatively efficient allocation. So, the sum of valuation also remains same this is 0. Payment received by agent 2 is what?

What is the sum of valuations of all the agents in this particular allocation except pay agent 2? So, if you forget edge 2, then the sum of valuations is minus 5 minus 2; that means, minus 5 minus 2 minus; now, in the absence of in the absence of agent 2. So, if the absence of edge A to t there is only one path s to B to t and the sum of valuations is minus 5 minus 10.

So, this is 8. So, payment received by agent 2 is 8 what is payment received by agent 3? So, sum of valuations of all the players in this particular path S to B to A to t except S to B while we are calculating payment for player 3, we will not add the valuations of valuation of agent 3. So, I will add 2 and 7. So, minus 2 minus 7 minus now, in the absence of player 3.

In the absence of player 3 there is only one path S to A to t. So, if in S to A to t the sum of valuations is minus 10 minus 7 is also 8. Payment received by agent 4, the valuation of this particular allocation 0, 1, 1, 0, 1 to agent 4 is 0; so, the sum of valuations of all the agents minus 14 minus minus 14, which is 0. Payment received by agent 5 ok what is the sum of valuations of all the items except agent 5? It is 5 plus 7 this particular rate path 5 plus 7 is minus 5 minus 7 minus in the absence of agent 5.

If agent 5 is not there then B to t edge is not there then we have two options S to A to t and S to B to t and the low cost one is S to B to t. So, in the absence of agent 2 any allocatively efficient allocation rule will pick S to B to t and its cost is sum of valuations is minus 5 minus 10 which is 3 ok. Now, let us. So, which path is chosen? S to B to A to t and player 2 gets 8, player 3 gets 8, player 5 gets 3, player 1 and player 4 do not get anything.

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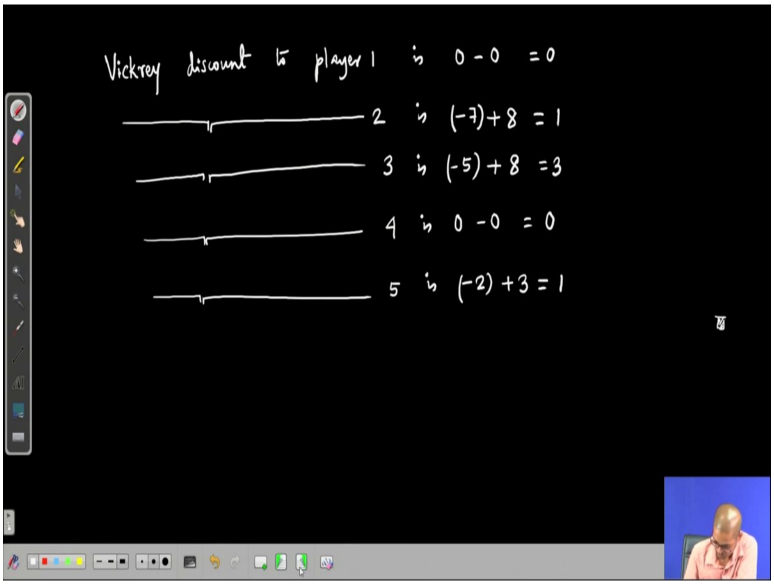

What is Vickrey discount? Vickrey discount to player 1 is it values this allocation to be 0 and it pays 0. So, it is 0 minus 0 which is 0. Vickrey discount to player 2 what is the Vickrey discount to player 2? How much player 2 values this particular allocation? What is player 2? Player 2 is this particular edge A to t and its cost is 7. So, this valuation is minus 7. So, minus 7 and it is getting paid 8; so, plus 8.

So, it is receiving 8, but its cost is only 7. So, it Vickrey discount is 1. Vickrey discount to player 3 is let us see player 3 is S to B its valuation is minus 5 and it gets 8. So, Vickrey discount is 3. Vickrey discount to player 4 is player 4 does not get anything is not paying anything and its valuation of this allocation is 0. So, it is 0 minus 0, 0 Vickrey discount to player 5 is, its valuation is minus 2, 5 the player 5 is the B to t edge valuation is minus 2 and it is getting paid 3. So, it is 1 Vickrey discount is 1 ok.

So, this concludes the our example and we will see some more examples in the assignments and tutorials. Now, let us ask so, ok. So, this Groves mechanism is very good it provides sufficient condition for it provides a sufficient condition for designing dominant strategy incentive compatible mechanisms in quasi environment.

Namely, if you give an any allocation rule, which is allocatively efficient then I can come up with a payment rule and it is much more general the grapes payment Groves payment rule is very general which makes the allocatively efficient allocation rule dominant strategy incentive compatible mechanism good.

But the question is that does there exist any other allocation rule? For example, suppose I give you an allocation rule, which is not allocatively efficient still is it possible to combine this allocation rule with a suitable payment rule which makes this entire mechanism dominant strategy incentive compatible.

Let me repeat is it possible or does there exist any allocation rule which is not allocatively efficient, but still it is possible to design a payment rule, which when combined with the allocation rule the resulting mechanism that we get is dominant strategy incentive compatible. And the answer is yes, it is possible and that is our topic of discussion next these are called weighted VCG.

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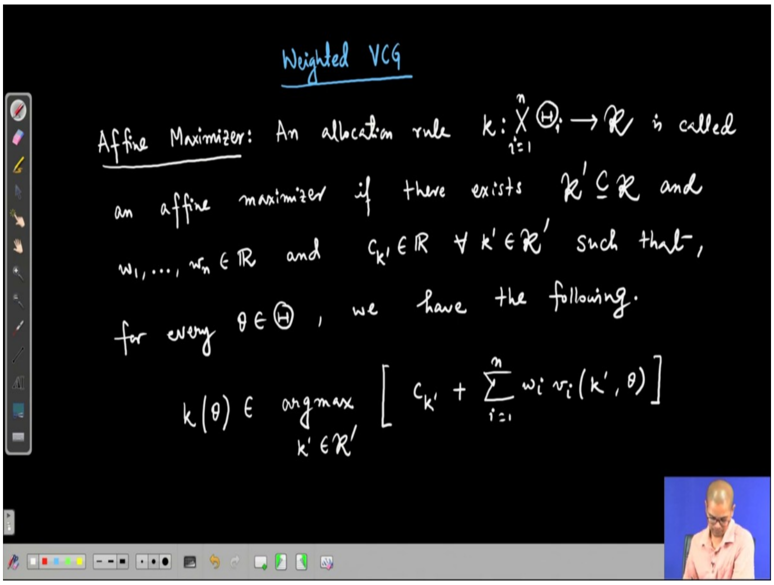

So, what are the mechanisms? What are the allocation rules which are implementable? And which is a generalization of allocatively efficient rule they are called affine maximizers an allocation rule k from $\times_{i=1}^{n} \Theta_i \rightarrow K$ is called an affine maximizer if there exists a subset of allocations $K' \subseteq K$ and constants $w_1, \ldots, w_n \in \mathbb{R}$ real constants and there are constants for each allocation $k' \in K'$.

Such that for every type profile theta in Θ we have the following; what is $k(\theta)$? k is the allocation rule $k(\theta)$ is the allocation chosen by the allocation rule k on the type at the type profile θ this is $argmax_{l'K'}\left[c_{k'} + \sum_{i=1}^{n} w_i v_i(k', \theta)\right]$. So, as you can see this is a generalization of affine this is generalization of allocative efficiency.

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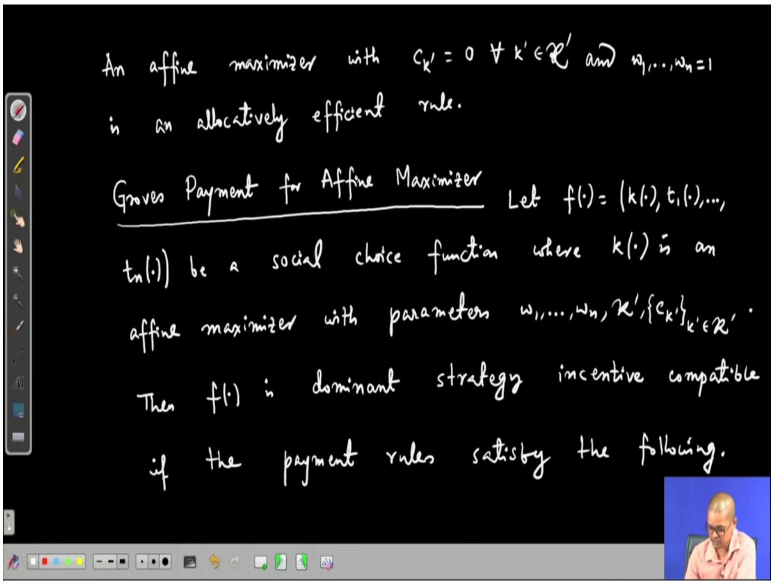

So, an affine maximizer with $c_{k'} = 0$ for all $k' \in K'$ and the weights are all $w_1, \ldots, w_n = 1$ is nothing but is an allocatively efficient rule. So, this is a generalization of allocative efficiency. Now, it is easy to generalize the Groves payment structure appropriately so, that we will study next.

So, Groves payment structure Grove's payment for affine maximizer. So, let f equal to $(k, t_1, \ldots, t_n)$ be a social choice function where, the allocation rule k is an affine maximizer with parameters $w_1, \ldots, w_n$ and $c_{k'}, k' \in K'$. These are the parameters then f is dominant strategy incentive compatible if the payment rules satisfy the following, which is a generalization of Grove's payment formula.

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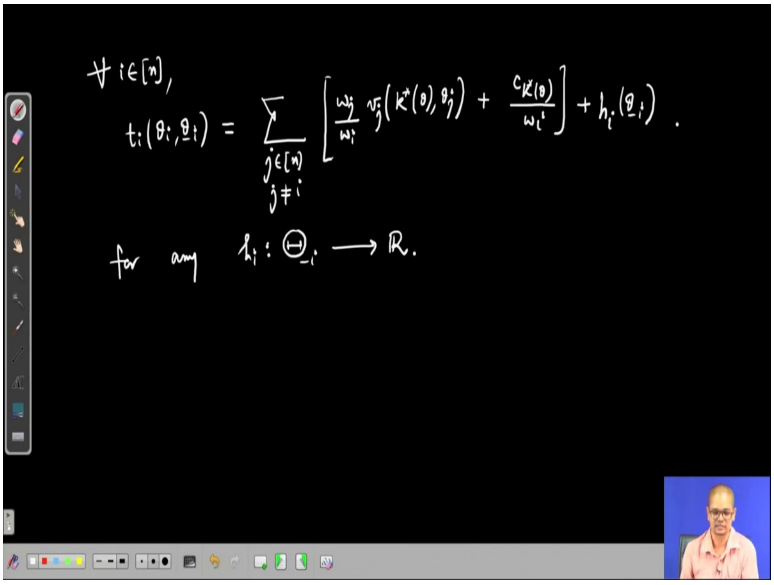

For all i in n, $t_i(\theta_i, \theta_{-i})$ is same as Groves payment structure we sum over all players except i and we sum the valuations $v_j(k^*(\theta), \theta_j)$, but because of those weights we scale these thing appropriately this is $\frac{w_j}{w_{¿}}$i plus $\frac{c_{k^*(\theta)}}{w_i}$ plus as usual in Groves $h_i(\theta_{-i})$ for any $h_i : \Theta_{-i} \rightarrow \mathbb{R}$. So, we will see the proof of this theorem in the next class.