Algorithmic Game Theory

Prof. Palash Dey
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 46
Ex-Post Efficiency

Welcome. So, in the last class we have started studying Quasi-Linear Environment and we have started studying various properties of quasi-linear environment like allocatively efficiency and strict budget balanced and we will continue that in today's lecture.

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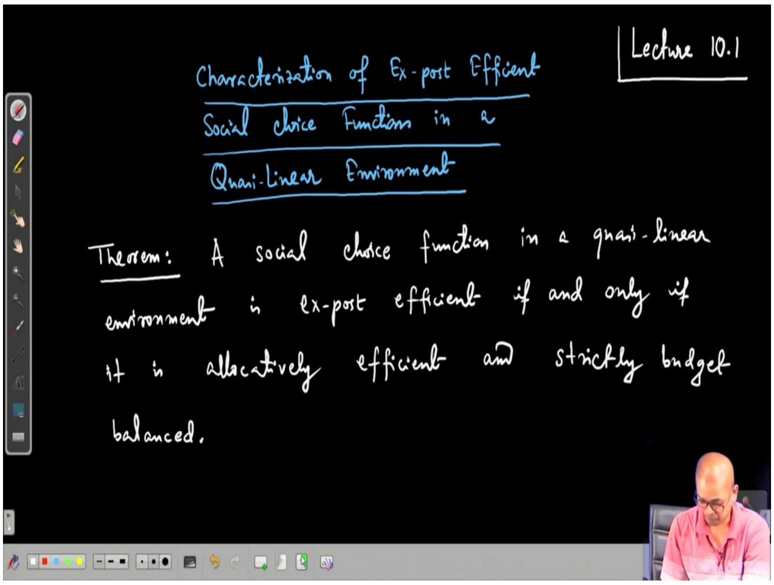

So, our first and most important result of today's class is characterization of ex-post efficient social choice functions characterization of ex-post efficient social choice functions in the in a quasi-linear environment. So, that is our main theorem today we will state and prove theorem.

That a social choice function in a quasi-linear environment is ex-post efficient if and only if and only if it is allocatively efficient and strictly budget balanced. You see that this quasi-linear environment allows us to study ex-post efficiency in terms of two in terms of properties of individual parts of the social choice function; that means, one property of allocation rule which is allocatively efficient efficiency.

And another property is for the payment rule which is strictly budget balanced. So, it makes the design and study of ex-post efficient social choice functions much more modular. Proof.

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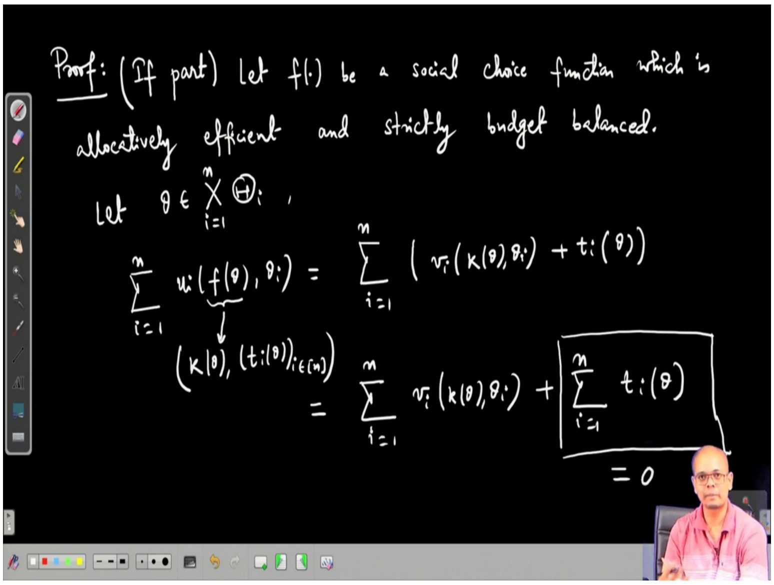

The proof is although simple enough. So, for the if direction. If part means that suppose there is a social choice function which is allocative efficient and strictly budget balanced, we need to show that this is ex-post efficient. So, let f be a social choice function which is allocatively efficient and strictly budget balanced.

So, we need to show that it is ex-post efficient. That means what? That means, for every type profile theta the outcome chosen by the social choice function is pareto optimal. There is no other outcome which makes every player as happy as this outcome and there exist at least one player who is strictly more happier there is no such outcome.

So, let us pick any type profile theta. And we need to show that this is ex-post efficient. So, let us find out what is summation of utilities $u_i(f(\theta))$. $f(\theta)$ is the outcome chosen by social choice function $f((\theta_i)_{i\in[n]})$. Now what is utility? Utility has two parts it has valuation $v_i(f(\theta), \theta_i) + t_i(\theta)$, i equal to 1 to n. Now this $f(\theta)$ actually has two parts.

First is payment $k(\theta)$ and this first is allocation and this payments $t_i(\theta) i \in [n]$. And this valuation depends only on the allocation. So, $v_i(k(\theta))$. Now because it is. So, now, we distribute this $\sum_{i=1}^{n} v_i(k(\theta),\theta_i)+\sum_{i=1}^{n} t_i(\theta)$. Now because this social choice function is strongly budget balanced this part is 0, summation of payments is 0.

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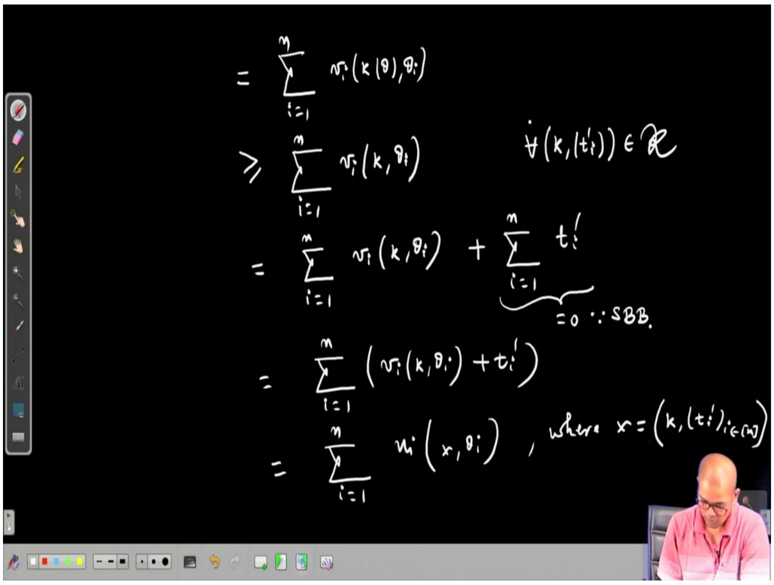

So, this is $\sum_{i=1}^{n} v_i(k(\theta),\theta_i)$. Now because this is this allocation rule is allocatively efficient k is allocatively efficient allocation rule we have this is greater than equal to. If I look at any other outcome and I look at the allocation part of that outcome.

So, we have this is 1 to n $v_i(k,\theta_i)$. This holds for all outcome $(k,(t_i'))\in K$. Now this we can write as s $\sum_{i=1}^{n} v_i(k(\theta),\theta_i)+\sum_{i=1}^{n} t_i'$. Why? Because this social choice function is strictly budget balance. So, if I pick any outcome and if I look at the payment parts of that outcome their sum is 0. So, $\sum_{i=0}^{n} t_i'=0$.

So, this term is 0. So, I am adding 0 here basically. Since we have strict budget balanced, but now this is $\sum_{i=1}^{n} v_i(k(\theta),\theta_i)+\sum_{i=1}^{n} t_i'$. This is $\sum_{i=1}^{n} u_i(x,\theta_i)$ where x is $(k,t_i')i\in[n]$.

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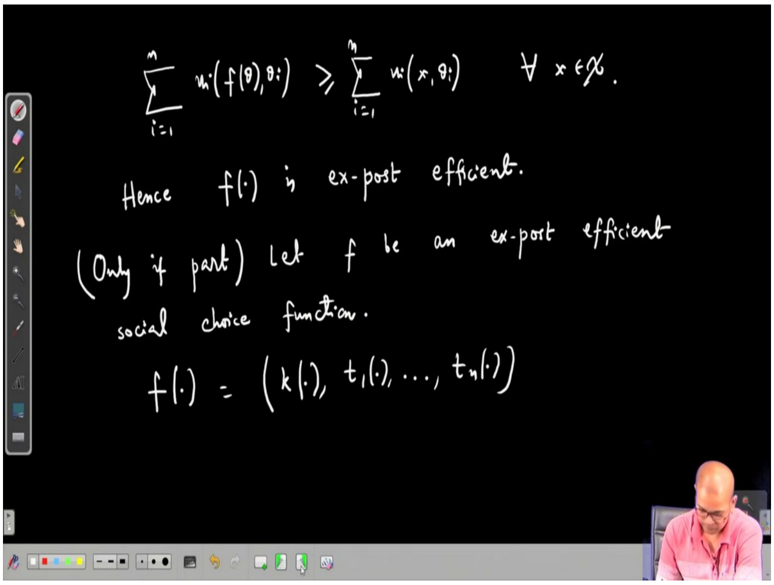

So, what we have? We have that $\sum_{i=1}^{n} u_i(f(\theta),\theta_i) \geq \sum_{i=1}^{n} u_i(f(\theta),\theta_i)$. This holds for all outcome $x \in X$. This is exactly the condition for ex-post efficiency indeed. So, suppose this condition is true, then f must be ex-post efficient.

Because if f is not exposed efficient then there exists an outcome say x where every players utility is as high as its utility with $f(\theta)$. But there exist at least one player whose utility is strictly more. So, that means, that the sum of utilities will be strictly more at x compared to at $f(\theta)$ which is not the case. Hence f is ex-post efficient. This concludes the first part of the proof the if part. Now the only if part; only if part. Let f be if let f be an ex-post efficient social choice function.

We will show that the if has two parts one is allocation rule another is payment rule. The allocation rule is allocatively efficient and the payment rule is strong is strictly budget balanced. So, the social choice function f looks like this is $(k, t_1, \ldots, t_n)$ these functions. Now we will first proof allocative efficiency of k.

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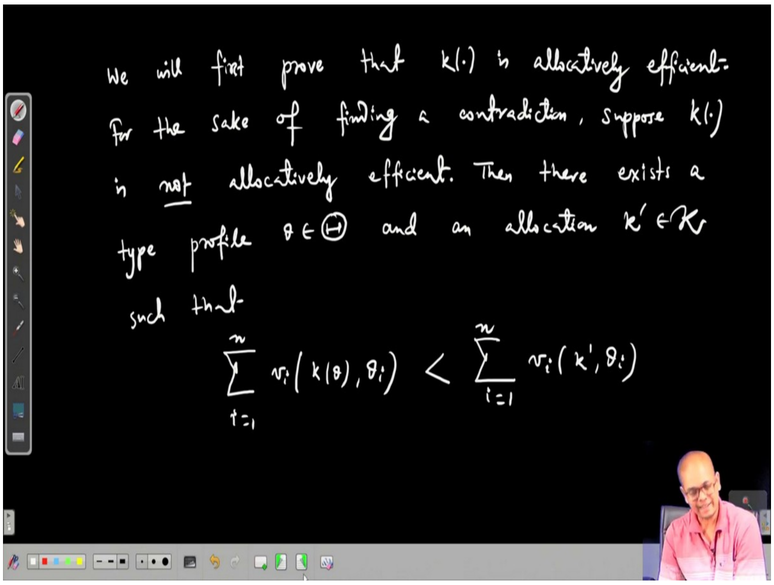

We will first prove that k is allocatively efficient. So, suppose not. Suppose this is not allocatively efficient. So, suppose not. So, for the sake of finding a contradiction, suppose k is not allocatively efficient. Then what will happen? Then there exists a type profile and an allocation $k^{'}$ in set of allocation K such that the $\sum_{i=1}^{n} v_i(k(\theta),\theta_i) < \sum_{i=1}^{n} v_i(k^{'},\theta_i)$.

So, if that if this happens then we will show that the social choice function f we begin with is not ex-post efficient.

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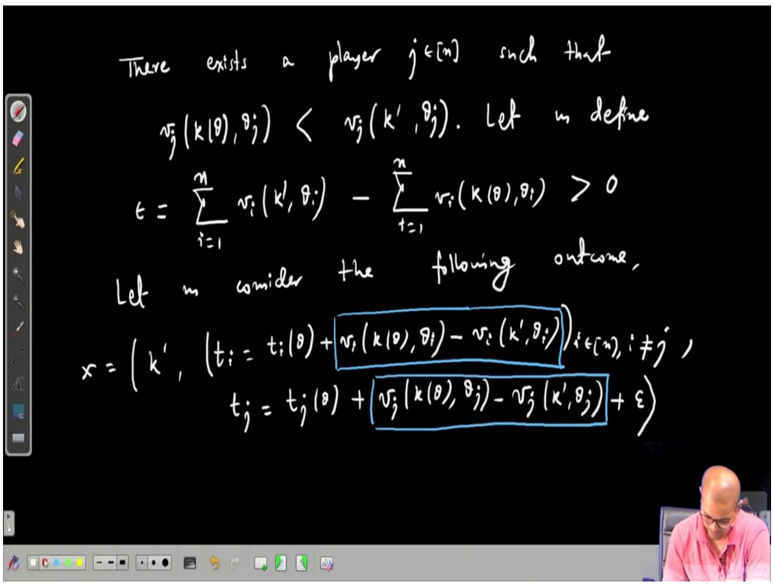

So, how will you show that? Then there exists an there exists a player $j \in [n]$ such that the valuation of player j in this. Valuation of this of player j $v_j(k, \theta_j) < v_j(k^{'}, \theta_j)$, ok. Now let us define ϵ to be the difference of valuation sum of valuations of all the players in the allocation $k(\theta)$ versus $k^{'}$.

So, $\sum_{i=1}^{n} v_i(k, \theta_i) - \sum_{i=1}^{n} v_i(k(\theta), \theta_i)$ this is a strictly positive term. Now, let us consider the following outcome. Let us consider the following outcome. What outcome takes? The allocation is $k^{'}$ and the payment is $t_i(\theta)$ and we adjust the change in valuation we compensate it here.

That means, that is $t_i(\theta) + v_i(k(\theta), \theta_i) - v_i(k^{'}, \theta_i)$. This is for all player $i \in [n], i \neq j$. And for i-th player for j-th player $t_j = t_j(\theta) + v_j(k(\theta), \theta_j) - v_j(k^{'}, \theta_j) + \epsilon$ plus this extra advantage epsilon you give it to the jth player.

Now the what is the idea? The idea is that you know we are changing the allocation from $k(\theta)$ to $k^{'}$ and this is the loss of valuation that player i suffers from $k(\theta)$ to $k^{'}$ by changing from $k(\theta)$ to $k^{'}$. And that much we are compensating him in the payment. That we are doing for each player plus the extra surplus in sum of valuations we are supplying to player j. That is on top of compensating for the loss of valuation for player j.

So, all the players except j they their utility remains same on the other hand player js utility is strictly more in the outcome x. And hence player j strictly likes this outcome x than $f(\theta)$ on the other hand all the other players are indifferent between x and $f(\theta)$. In particular x pareto dominates $f(\theta)$. So, let us write this thing. So, but first of all whether it satisfies the budget balance condition or not that we need to verify.

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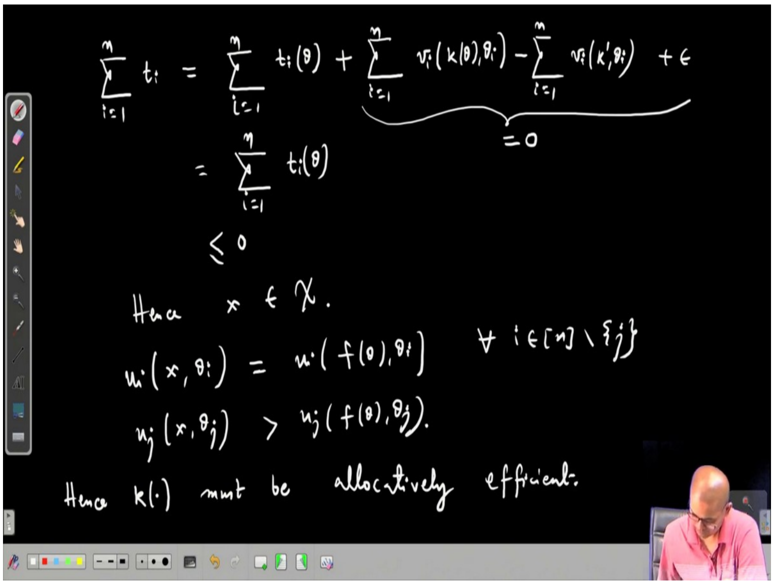

So, $\sum_{i=1}^{n} t_i$. Because otherwise x may not be a valid outcome. For x to be a valid outcome this t_i is must satisfy the weekly budget balanced condition. This is $\sum_{i=1}^{n} t_i(\theta)+\sum_{i=1}^{n}(v_i(k(\theta),\theta_i)-v_i(k',\theta_i))+\epsilon \quad \epsilon=\sum_{i=1}^{n} v_i(k',\theta_i)-\sum_{i=1}^{n} v_i(k(\theta),\theta_i)$.

So, this part is 0. So, this is $\sum_{i=1}^{n} t_i(\theta_i)$. This is less than equal to 0. So, this payment satisfy weekly budget balance condition. And hence x belongs to the set of all outcomes. And what we have observed is that utility of any player i in the outcome $u_i(x,\theta_i)$ it is same with $u_i(f(\theta),\theta_i)$.

This holds for all player $i \in [n]$ except j and for j utility is strictly better. Hence this contradicts our assumption that the social choice function that we began with is a allocatively efficient social choice function. Hence ex-post efficient social choice function. Hence the allocation function k must be allocatively efficient. Next, we prove a strongly budget balance condition so.

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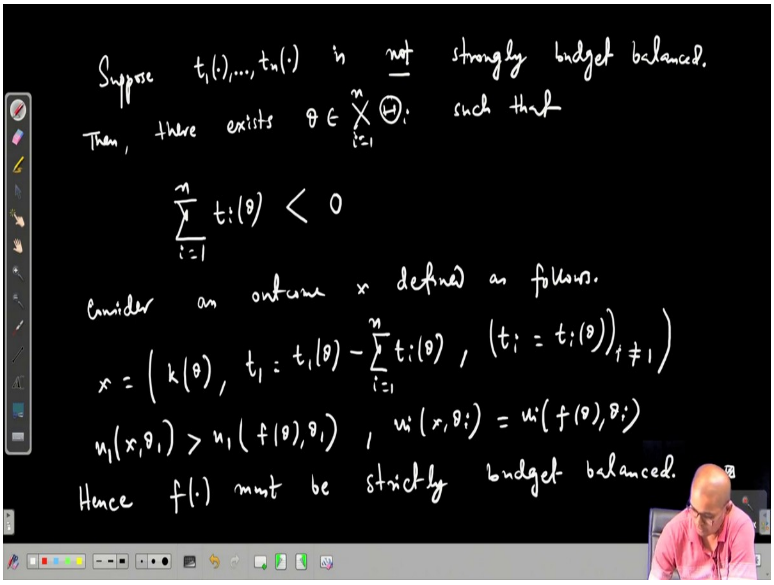

So, suppose this payment functions $t_1, \ldots, t_n$ is not strongly budget balanced. Then, there exists a type profile $(\theta_i)_{i \in [n]}$ such that $\sum_{i=1}^{n} t_i(\theta_i) < 0$. These payment rules are always weakly budget balance. So, this has to be always less than equal to 0, but if it is not strongly budget balance; that means, there exist some type where this is strictly less than 0.

Now what we do is that. We now consider an outcome an outcome x defined as follows. What is x? The allocation let it remain same. But the surplus money let us give it to someone say player one. So, say $t_1 = t_1(\theta) - \sum_{i=1}^{n} t_i(\theta)$

So, player i is getting strictly more money. Because $\sum_{i=1}^{n} t_i(\theta_i) < 0$. For other players it remains same. $t_i = t_i(\theta), i \neq 1$. So, clearly all the players for all the players the utility remains same, but only for player one. So, $u_1(x, \theta_1) > u_1(f(\theta), \theta_1)$.

Simply because the allocation remains same hence the valuation remains same, but it is getting strictly more money the player one. For other players their valuation and payment remains exactly same. $u_i(x, \theta_i) = u_i(f(\theta), \theta_i)$. So, but this contradicts our assumption that the social choice function f is ex-post efficient.

Because I am able to come up with another outcome which makes which keeps all player as a p as $f(\theta)$, but there is one player one player namely player one who is strictly more happy happier which is which contradicts our assumption that f is ex-post efficient. Hence f must be strongly budget balanced strictly budget balanced. This which concludes the proof. So, we will stop here today.