Algorithmic Game Theory
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Lecture - 45
Quasilinear Environment

Welcome. So, in the last lecture we have studied Gibbard-Satterthwaite theorem which is a cornerstone impossibility result of game theory and social choice theory.

And it shows that almost all social choice functions are not implementable in dominant strategy equilibrium and the only exceptions are if we have two candidates two outcomes then the then we have rules. But if we have more than two outcomes then there is there is only dictatorship social choice functions which are implementable if it is if it is unanimous.

So, at the end of last lecture we have studied we have discussed various directions.

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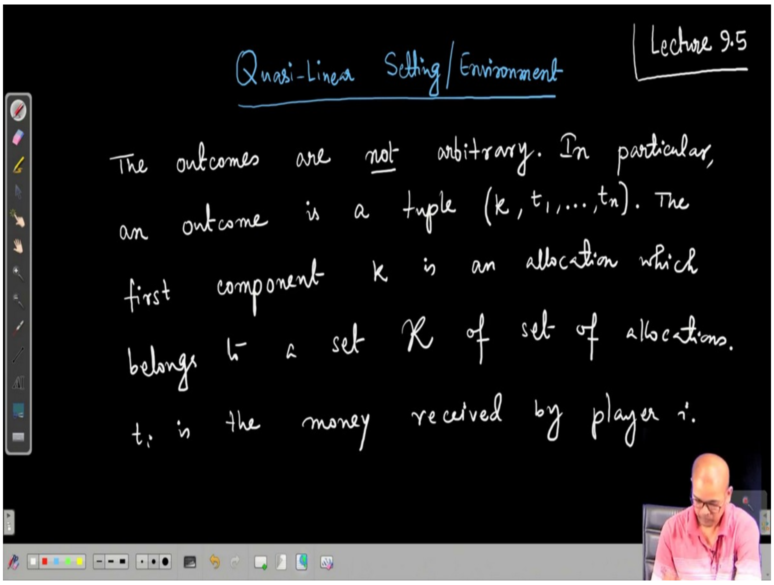

And from today we will start excluding one important direction to overcome or circumvent the impossibility result of Gibbard-Satterthwaite theorem and that is called Quasilinear Setting or it is also called Quasilinear Environment.

So, what is the setting? First of all the outcomes are not arbitrary, means an outcome in particular outcome has a certain structure in particular an outcome is a tuple $(k, t_1, \ldots, t_n)$. So, $(t_1, \ldots, t_n)$ where n is the number of players and this k the first component k is an allocation which belongs to set cal K of set of allocations.

So, a model example is a auction scenario where the first component k denotes which player gets the item or gets to sell the item the allocation and $(t_1, \ldots, t_n)$ is the is the t_i is the money received by player i ok.

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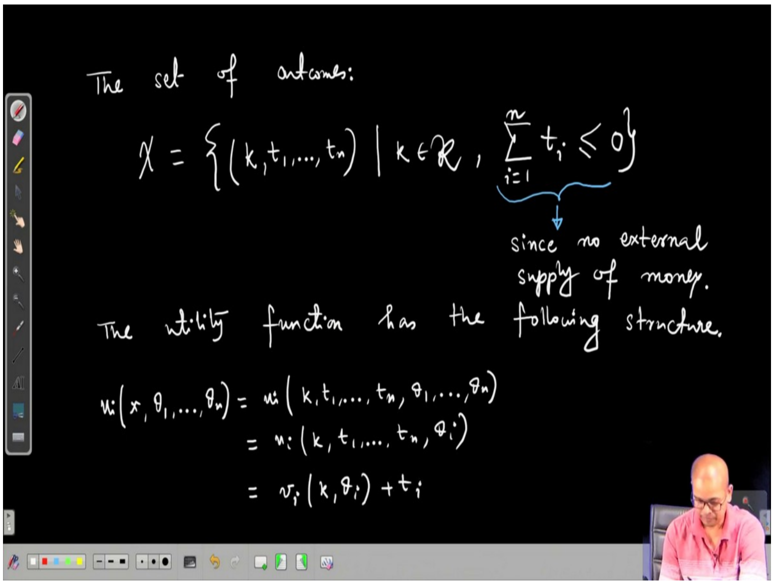

So, the set of outcomes in a quasilinear environment is this x is k the first component is allocation and then $(t_1, \ldots, t_n)$, such that the allocation k belongs to cal K, the set of all allocations and these $(t_1, \ldots, t_n)$ are the money received by individual players, but there is no external supply of money.

So, $\sum_{i=1}^{n} t_i \leq 0$ a mechanism can generate money, but because there is no external supply of money, $\sum_{i=1}^{n} t_i$ sum of total money received should be less than equal to 0 this is since no external supply of money ok. And how does the utility look like? The utility function utility function has the following structure.

So, utility of player i in a particular outcome x when the types are $(t_1, \ldots, t_n)$ this is u i this x outcome has a allocation is a tuple of allocation and payment received. So, it is k then $(t_1,\ldots,t_n)$ and of course, we have $(\theta_1,\ldots,\theta_n)$, first of all the outcome the utility of player i depends only on the type of player 1 and the outcome it does not depend on the type of other players is θ_i and now each player i has a valuation for the allocation and that depends on its type also, so $v_i(k,\theta_i)+t_i$.

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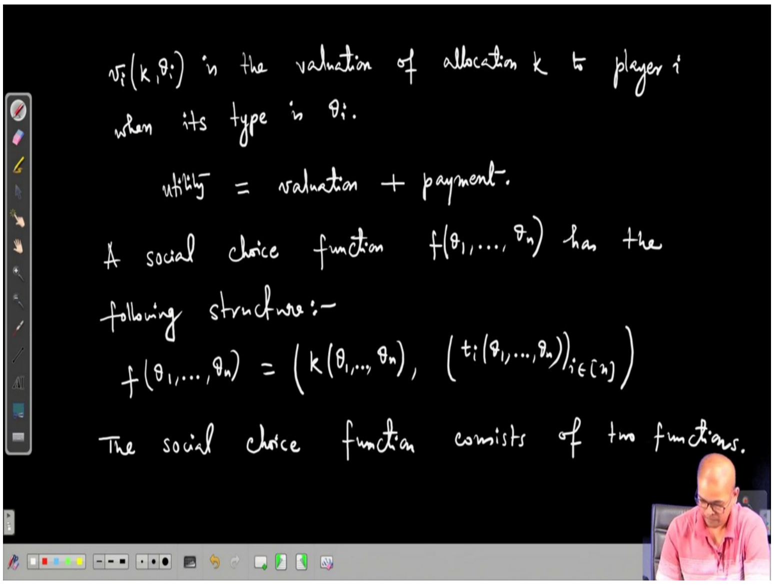

So, $v_i(k,\theta_i)$ is the valuation of allocation k to player i when its type is θ_i. So, utility is valuation plus payment. So, that is the structure of utility function in the quasilinear setting, utility is equal to valuation plus payment ok.

So, a social choice function $f(\theta_1,\ldots,\theta_n)$ has the following structure has the following structure in the quasilinear environment. What structure? What is $f(\theta_1,\ldots,\theta_n)$? Recall, a social choice function maps type profiles to outcome and outcome in a in a quasilinear environment looks like a allocation and payment this tuple.

So, this will be an allocation which depends on of course, $(\theta_1,\ldots,\theta_n)$ and payment which of course, also depends on the type profile $(\theta_1,\ldots,\theta_n)$. The first part is called allocation function and the second part is called payment function. So, the social choice function consists of two functions in a quasilinear environment two functions, what are they?

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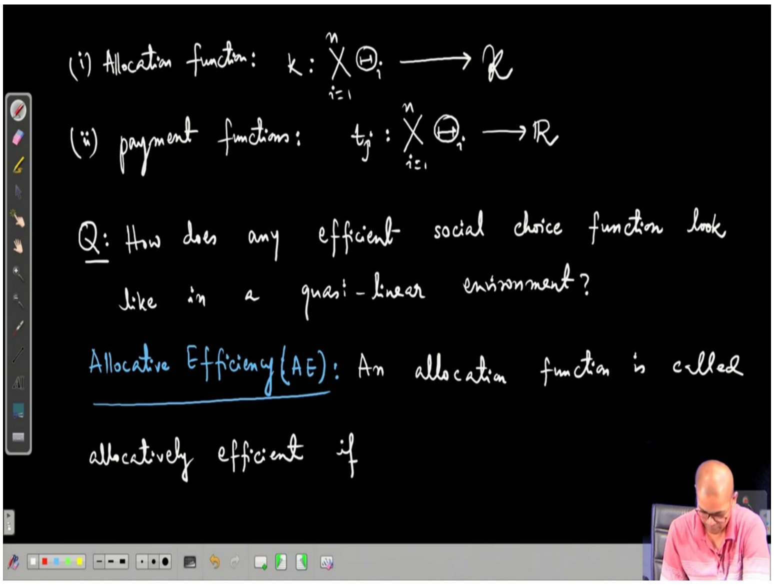

The first one is called an allocation function. So, this allocation function k is a mapping from set of all type profiles to the set of all outcomes K. And the payment function we have n payment functions. So, payment functions t_i is or t_j is function from type profiles to real numbers ok.

Now, recall, we have studied various properties of social choice functions and the first one and most important one is the efficiency or Pareto efficiency. So, how does Pareto efficient social choice functions look like in the quasilinear environment, that is our topic of investigation next.

So, question, how does an any efficient social choice function look like in a quasilinear environment? How does they look like? So, because social choice function is a combination of two functions; one is allocation function and payment function, we will characterize efficiency of social choice functions in terms of properties of allocation function and payment function in a quasilinear environment.

So, the first property that we study is for allocation function and that property is called allocative efficiency it is in short it is called AE. An allocation function is called allocatively efficient.

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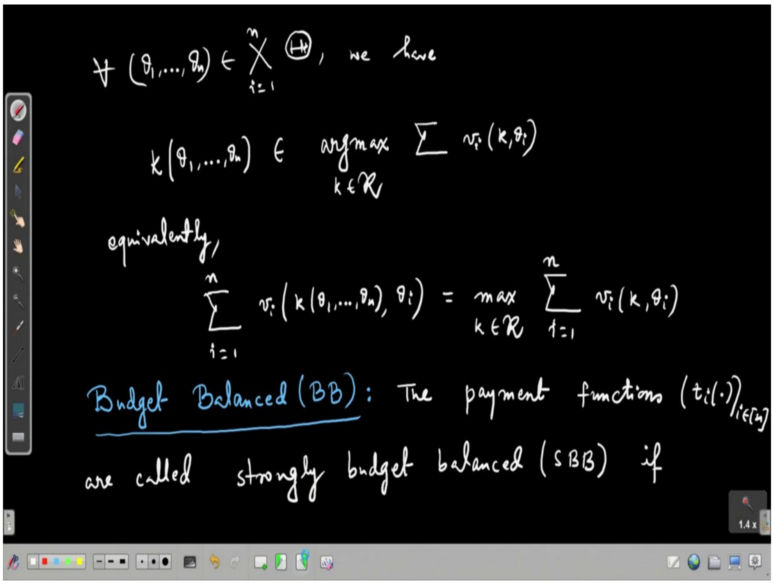

If for all type profile $(\theta_1,\ldots,\theta_n)$ in set of all type profiles $\times_{i=1}^{n}\Theta_i$, we have the allocation chosen by the allocation function theta on $k(\theta_1,\ldots,\theta_n)$ maximizes sum of valuations ok.

So, over all allocations you the allocatively efficient allocation rule picks the allocation which maximizes sum of valuations. So, alternatively or equivalently $\sum_{i=1}^{n} v_i(k(\theta_1,\ldots,\theta_n),\theta_i)$ this is $max_{k\in K}\sum_{i=1}^{n} v¿i(k,\theta_i)$ ok. So, this is the property of allocation function.

Now, next we will we will study a property of payment function and that is called budget balanced budget balanced BB or the payment functions t i are called strongly budget balanced SBB or simply budget balance just SBB, if there is no surplus of money or demand of money for each type profile.

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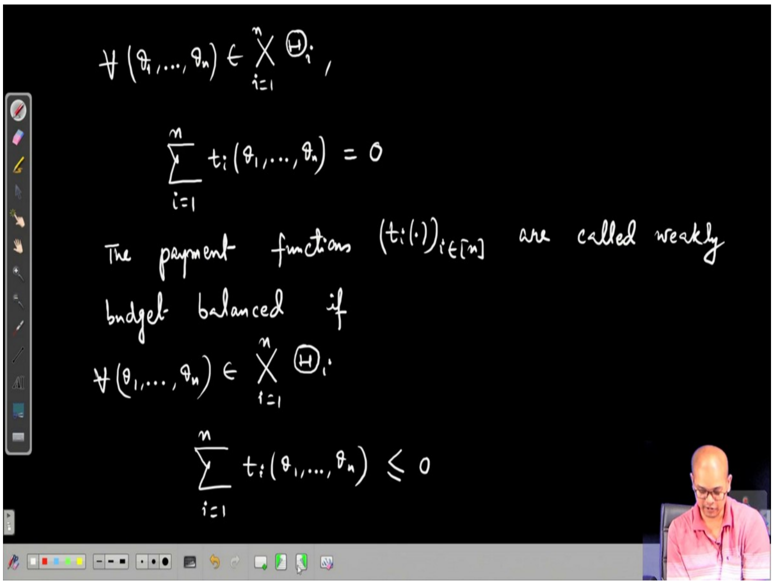

If for each type profile $(\theta_1,\ldots,\theta_n)$ $\sum_{i=1}^{n} t_i(\theta_1,\ldots,\theta_n)=0$, typically we assume only weak budget balanced property. So, that is the condition written in the set of all outcomes for quasilinear environment; that means, that the there can be surplus of money, but there should not be any shortage of money there is no external monetary supply.

So, let us write that the payment functions $(t_i(.))_{i\in[n]}$ are called weekly budget balanced if for all type profile $(\theta_1,\ldots,\theta_n) \times_{i=1}^{n} \Theta_i$ $\sum_{i=1}^{n} t_i(\theta_1,\ldots,\theta_n)\leq 0$.

So, there can be surplus of money, but there should not be any deficit of money t_i is the money received by player i. So, sum of money received must be less than equal to 0 if t_i is negative then player i will play will pay that much amount of money ok.

Now, with these two conditions we can characterize the non dictatorship functions and we. So, now, the first important observation is that there is no non dictatorship function in quasilinear environment.

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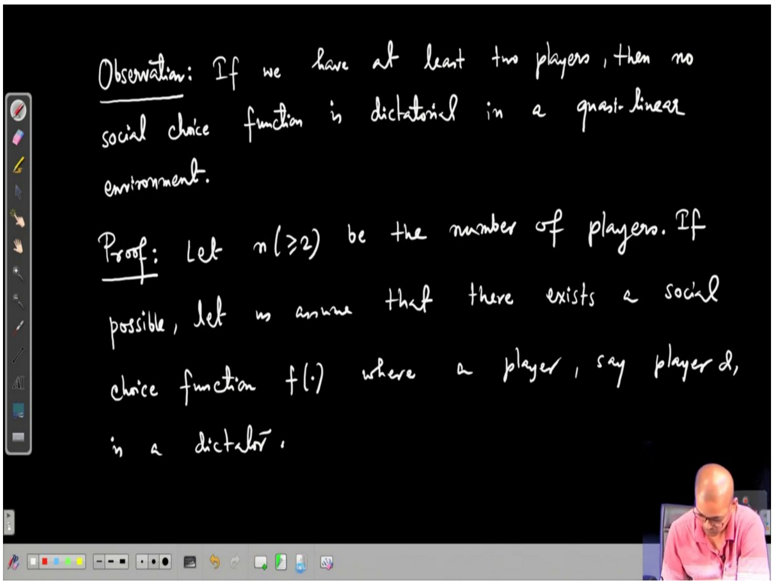

So, observation, the notion of dictatorship itself does not make sense in quasilinear environment, if we have at least two players, then no social choice function is dictatorial in a quasilinear environment. It is a grand it is a grand statement because you know the Gibbard-Satterthwaite theorem says that if you have at least three outcomes then all social choice functions that are implementable is in dominant strategy equilibrium are only dictatorship if it is unanimous.

But in quasilinear environment there we do not have any dictatorial function if we have at least two players and the proof is very simple proof. So, the high level idea is that you know there is no sort of best outcome for any player. So, if there is a detector you its outcome can be even improved by paying it more and charging the other player even more that is the idea.

So, let n which is greater than equal to 2 be the number of players, now for the sake of contradiction. So, let us assume if possible, let us assume that there exists a social choice function f where a player say, player d he is a dictator, we will contradict this assumption. So, let us see.

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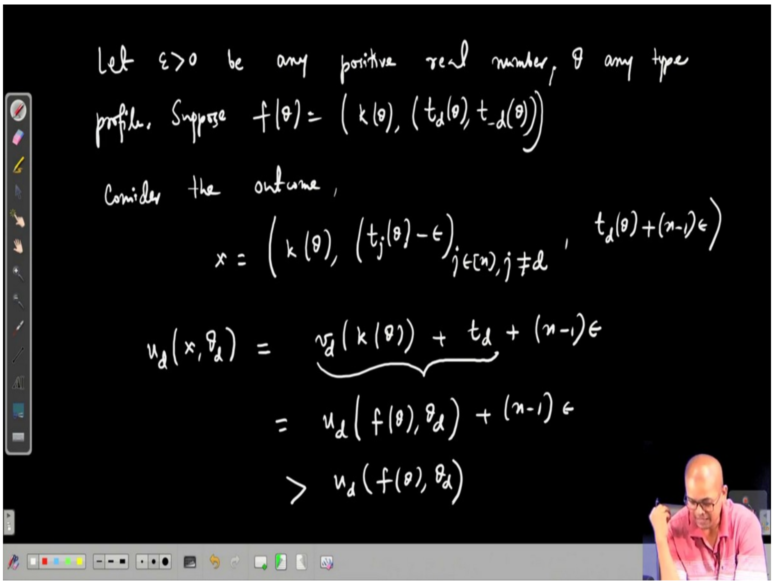

So, let speak any $\epsilon > 0$ be any positive real number and pick any θ_b and θ any type profile. So, suppose $f(\theta)$ suppose $f(\theta)$ is this $(k(\theta), t_d(\theta), t_{-d}(\theta))$. Now, what we can do is that we can charge epsilon more to each player and pay that to the player d the dictator.

So, consider the outcome x equal to the allocation let it remain same, but $t_j(\theta)$ you each player you take ϵ from there. So, $j \in [n]$ and $j \neq d$ and that much money you pay to the dictator $t_d(\theta) + (n-1)\epsilon$. Now, we have that u_d what is the utility of player d in the outcome x and its type remains θ_d its utility is valuation plus payment. So, this is $v_d(k(\theta), \theta_d)$ and payment is $t_d + (n-1)\epsilon$.

The first summation of first two terms is the utility of player i a player d in the outcome $f(\theta) + (n-1)\epsilon$. So, this is strictly more than $u_d(f(\theta), \theta_d)$ which is a contradiction. We are able to find another outcome x which gives strictly more utility than the outcome chosen by the social choice function f.

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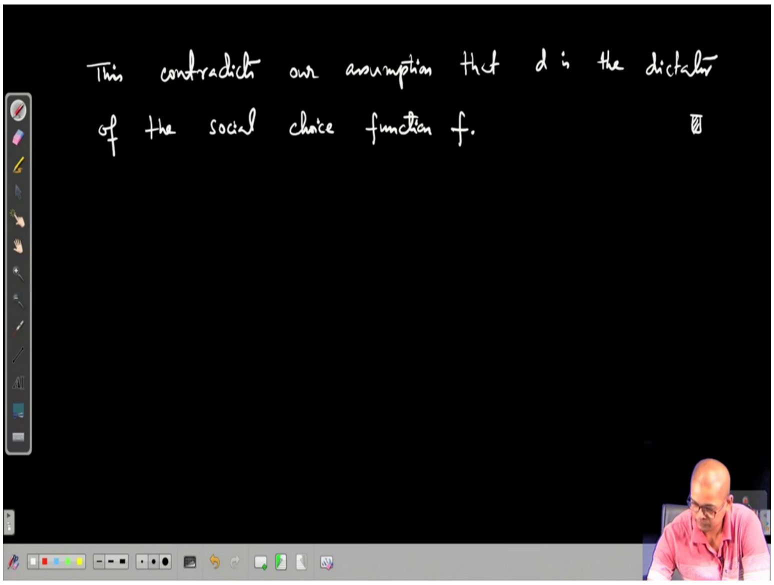

So, this contradicts this contradicts our assumption that d is the dictator of the social choice function f which completes the proof. So, we will end here today.